

FOURIER ANALYSIS TECHNIQUES

LEARNING GOALS

FOURIER SERIES

Fourier series permit the extension of steady state analysis to general periodic signal.

FOURIER TRANSFORM

Fourier transform allows us to extend the concepts of frequency domain to arbitrary non-periodic inputs



FOURIER SERIES

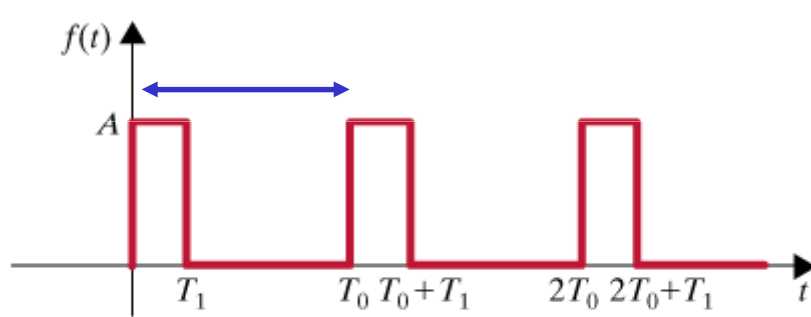
The Fourier series permits the representation of an arbitrary periodic signal as a sum of sinusoids or complex exponentials

Periodic signal

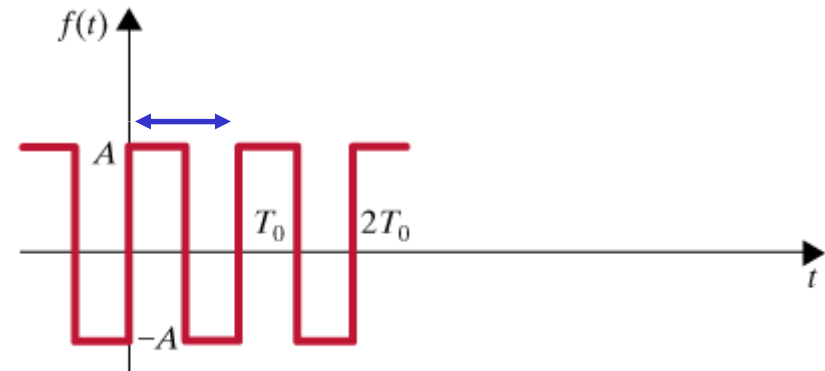
The signal $f(t)$ is periodic iff there exists $T > 0$ such that

$$f(t) = f(t+T), \forall t$$

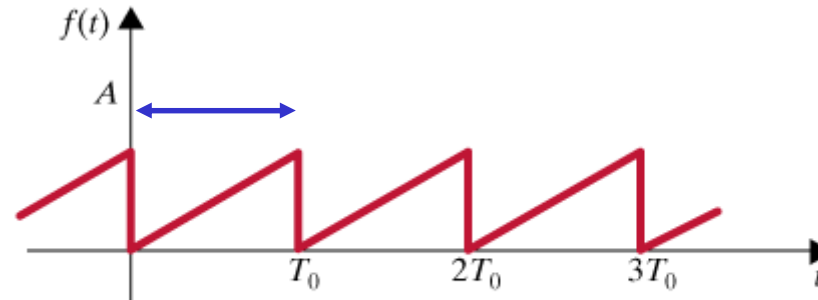
The *smallest* T that satisfies the previous condition is called the (fundamental) period of the signal



(a)



(b)



(c)



FOURIER SERIES RESULTS

If $f(t)$ is periodic, with period T_0 , then $f(t)$ can be expressed in one of the following equivalent forms

Cosine expansion

$$f(t) = a_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega_0 t + \theta_n) = a_0 + \sum_{n=1}^{\infty} \text{Re}[D_n \angle \theta_n e^{jn\omega_0 t}]$$

Phasor for n-th harmonic

$$\omega_0 = \frac{2\pi}{T_0}$$

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

Complex exponential expansion

$$D_n \angle \theta_n = 2c_n = a_n - jb_n$$

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

Trigonometric series

$$e^{j\alpha} = \cos \alpha + j \sin \alpha$$

$$e^{j\alpha} + e^{-j\alpha} = 2 \cos \alpha$$

$$e^{j\alpha} - e^{-j\alpha} = 2j \sin \alpha$$

$$c_0 = a_0$$

For $n > 0$

Relationship between exponential and trigonometric expansions

$$c_n e^{jn\omega_0 t} + c_{-n} e^{-jn\omega_0 t} = (c_n + c_{-n}) \cos n\omega_0 t + j(c_n - c_{-n}) \sin n\omega_0 t$$

$$a_n$$

$$b_n$$

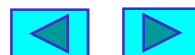
$$\Rightarrow 2c_n = a_n - jb_n$$

$$2c_{-n} = a_n + jb_n$$

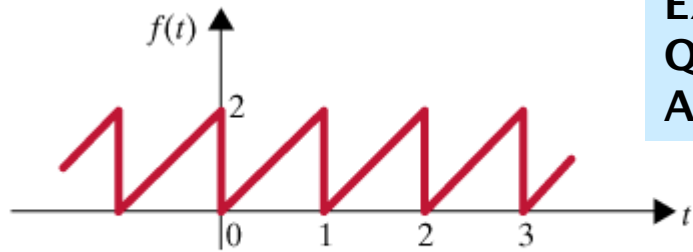
If $f(t)$ is real-valued then $c_{-n} = (c_n)^*$

GENERAL STRATEGY:

- . Approximate a periodic signal using a Fourier series
- . Analyze the network for each harmonic using phasors or complex exponentials
- . Use the superposition principle to determine the response to the periodic signal



Original Periodic Signal

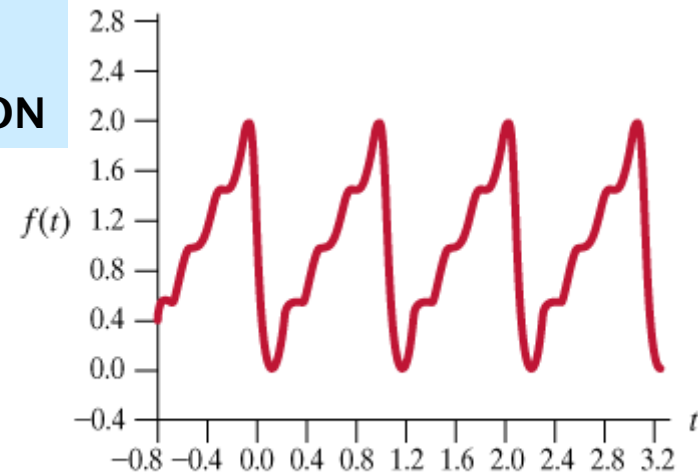


(a)

EXAMPLE OF
QUALITY OF
APPROXIMATION

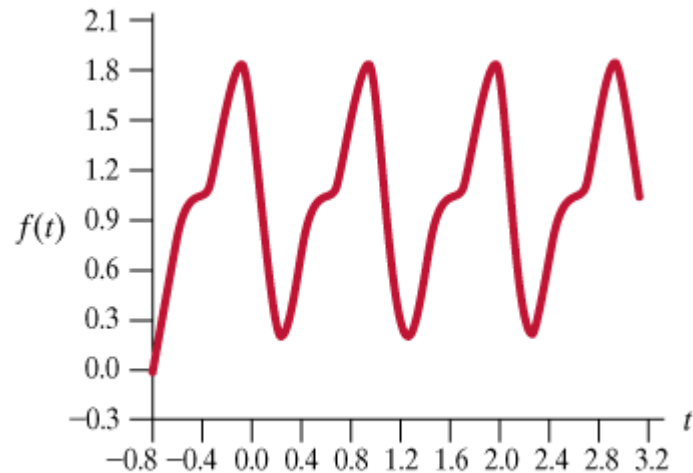
$$f_N(t) = \sum_{n=-N}^N c_n e^{jn\omega_0 t}$$

Approximation with 4 terms



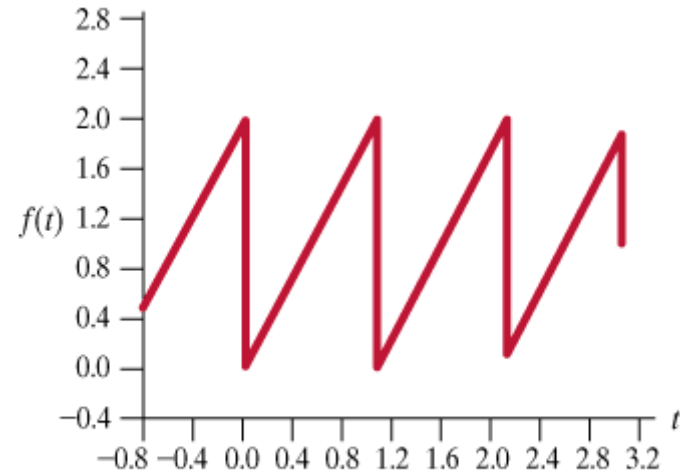
(c)

Approximation with 2 terms

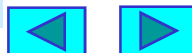


$$f_2(t) = a_0 + \sum_{n=1}^2 a_n \cos n\omega_0 t + \sum_{n=1}^2 b_n \sin n\omega_0 t$$

Approximation with 100 terms



$$a_0 + \sum_{n=1}^{100} a_n \cos n\omega_0 t + \sum_{n=1}^{100} b_n \sin n\omega_0 t$$



EXPONENTIAL FOURIER SERIES

Any “physically realizable” periodic signal, with period T_0 , can be represented over the interval $t_1 < t < t_1 + T_0$ by the expression

$$f(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t}; \quad \omega_0 = \frac{2\pi}{T_0}$$

The sum of exponential functions is always a continuous function. Hence, the right hand side is a continuous function.

Technically, one requires the signal, $f(t)$, to be at least piecewise continuous. In that case, the equality does not hold at the points where the signal is discontinuous

Computation of the exponential Fourier series coefficients

$$\int_{t_1}^{t_1+T_0} f(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t} \times e^{-jk\omega_0 t}$$

$$\int_{t_1}^{t_1+T_0} e^{j(n-k)\omega_0 t} dt = \begin{cases} 0 & \text{for } n \neq k \\ T_0 & \text{for } n = k \end{cases}$$

$$\int_{t_1}^{t_1+T_0} f(t) e^{-jk\omega_0 t} dt = \int_{t_1}^{t_1+T_0} \left(\sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t} \right) e^{-jk\omega_0 t} dt$$

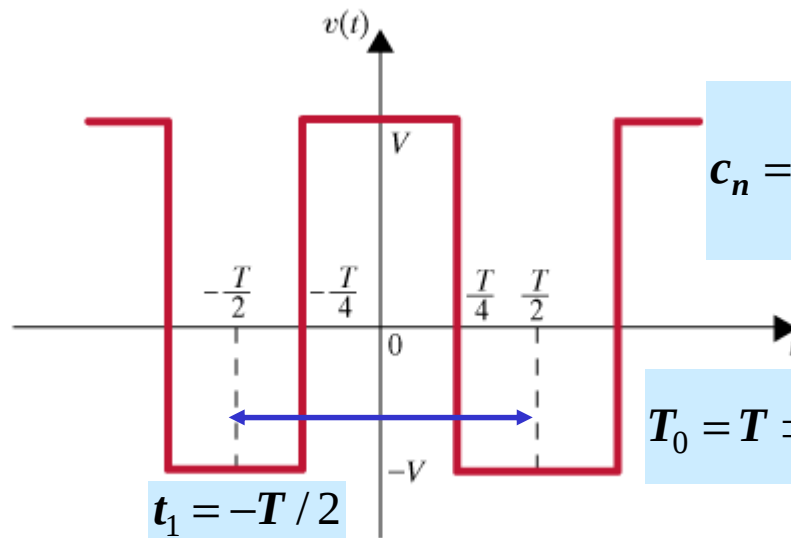
$$c_k = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} f(t) e^{-jk\omega_0 t} dt$$

t_1 is arbitrary and can be chosen to make computations simpler



LEARNING EXAMPLE

Determine the exponential Fourier series



$$c_n = \begin{cases} 0 & n \text{ even} \\ \frac{2V}{\pi n} \sin \frac{n\pi}{2} & n \text{ odd} \end{cases}$$

$$T_0 = T \Rightarrow \omega_0 = \frac{2\pi}{T}$$

$$f(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t}; \quad \omega_0 = \frac{2\pi}{T_0}$$

$$c_n = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} f(t) e^{-jn\omega_0 t} dt$$

A strategy :

1. Determine T_0 and ω_0
2. Select a convenient t_1
3. Do the integration

$$c_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} v(t) e^{-jn\omega_0 t} dt = \frac{1}{T} \int_{-\frac{T}{2}}^{-\frac{T}{4}} (-V) e^{-jn\omega_0 t} dt + \frac{1}{T} \int_{-\frac{T}{4}}^{\frac{T}{4}} V e^{-jn\omega_0 t} dt - \frac{1}{T} \int_{\frac{T}{4}}^{\frac{T}{2}} V e^{-jn\omega_0 t} dt$$

$$c_n = \frac{V}{T(jn\omega_0)} \left[e^{-jn\omega_0 t} \Big|_{-T/2}^{-T/4} - e^{-jn\omega_0 t} \Big|_{-T/4}^{T/4} + e^{-jn\omega_0 t} \Big|_{T/4}^{T/2} \right]$$

This is for $n \neq 0$!

$c_0 = 0$ in this case

$$c_n = \frac{V}{jTn\omega_0} \left[e^{-jn\omega_0 \left(-\frac{T}{4}\right)} - e^{-jn\omega_0 \left(-\frac{T}{2}\right)} - e^{-jn\omega_0 \left(\frac{T}{4}\right)} + e^{-jn\omega_0 \left(\frac{T}{2}\right)} + e^{-jn\omega_0 \left(\frac{T}{4}\right)} - e^{-jn\omega_0 \left(\frac{T}{2}\right)} \right]$$

$$c_n = \frac{V}{j2\pi n} \left[2e^{j\frac{n\pi}{2}} - 2e^{-j\frac{n\pi}{2}} + e^{-jn\pi} - e^{jn\pi} \right] = \frac{V}{2\pi n} \left[4\sin\left(\frac{n\pi}{2}\right) - 2\sin(n\pi) \right]$$

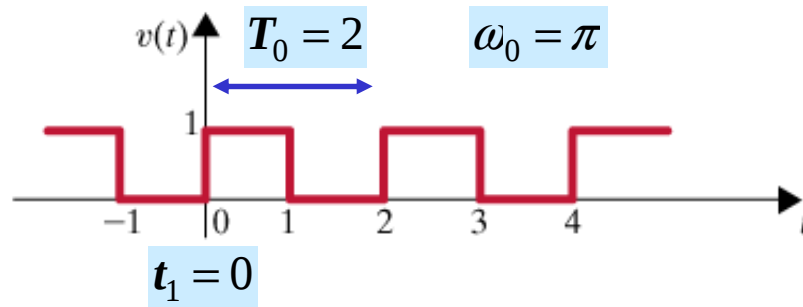
$$\frac{e^{j\alpha} - e^{-j\alpha}}{j} = 2\sin \alpha$$

$$T\omega_0 = 2\pi$$



LEARNING EXTENSION

Determine the exponential Fourier series



$$f(t) = \sum_{n=-\infty}^{n=\infty} c_n e^{jn\omega_0 t}; \quad \omega_0 = \frac{2\pi}{T_0}$$

$$c_n = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} f(t) e^{-jn\omega_0 t} dt$$

A strategy :

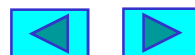
1. Determine T_0 and ω_0
2. Select a convenient t_1
3. Do the integration

$$c_n = \frac{1}{2} \int_0^2 v(t) dt = \frac{1}{2} \left[\int_0^1 e^{-jn\pi t} dt \right] = \frac{1}{2j\pi n} \left[-e^{-jn\pi t} \right]_0^1 = \frac{1 - e^{-jn\pi}}{2j\pi n};$$

For $n \neq 0$

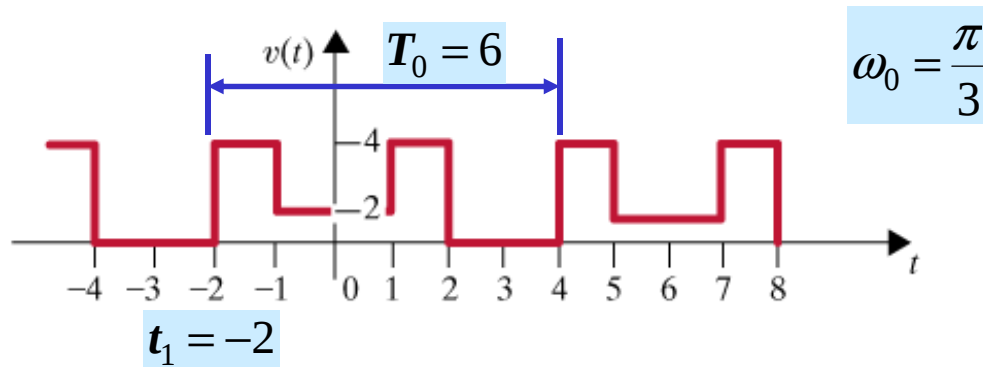
For $n = 0$

$$c_0 = \frac{1}{2}$$



LEARNING EXTENSION

Determine the exponential Fourier series



$$c_0 = \frac{1}{6} \int_{-2}^4 v(t) dt = 2$$

$$c_n = \frac{1}{6} \int_{-2}^4 v(t) dt = \frac{1}{6} \left[\int_{-2}^{-1} 4e^{-j\frac{n\pi}{3}t} dt + \int_{-1}^1 2e^{-j\frac{n\pi}{3}t} dt + \int_1^4 4e^{-j\frac{n\pi}{3}t} dt \right]$$

$$c_n = \frac{n\pi}{2j} \left[4e^{j\frac{2n\pi}{3}} - 4e^{j\frac{n\pi}{3}} + 2e^{j\frac{n\pi}{3}} - 2e^{-j\frac{n\pi}{3}} + 4e^{-j\frac{n\pi}{3}} - 4e^{-j\frac{2n\pi}{3}} \right]$$

$$c_n = n\pi \left[4\sin\frac{2\pi n}{3} - 2\sin\frac{\pi n}{3} \right]$$

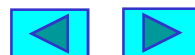
$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}; \quad \omega_0 = \frac{2\pi}{T_0}$$

$$c_n = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} f(t) e^{-jn\omega_0 t} dt$$

A strategy :

1. Determine T_0 and ω_0
2. Select a convenient t_1
3. Do the integration

$$\frac{e^{j\alpha} - e^{-j\alpha}}{j} = 2\sin\alpha$$



TRIGONOMETRIC FOURIER SERIES

$$f_2(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t$$

$$\omega_0 = \frac{2\pi}{T_0}$$

$$e^{j\alpha} = \cos \alpha + j \sin \alpha$$

$$e^{j\alpha} + e^{-j\alpha} = 2 \cos \alpha$$

$$e^{j\alpha} - e^{-j\alpha} = 2j \sin \alpha$$

$$c_0 = a_0$$

For $n > 0$

Relationship between exponential and trigonometric expansions

$$c_n e^{jn\omega_0 t} + c_{-n} e^{-jn\omega_0 t} = (c_n + c_{-n}) \cos n\omega_0 t + j(c_n - c_{-n}) \sin n\omega_0 t$$

$$a_n$$

$$b_n$$

$$\Rightarrow 2c_n = a_n - jb_n$$

$$2c_{-n} = a_n + jb_n$$

$$c_n = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} f(t) e^{-jn\omega_0 t} dt$$

$$a_0 = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} f(t) dt$$

$$a_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} f(t) \cos n\omega_0 t dt$$

$$b_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} f(t) \sin n\omega_0 t dt$$

If $f(t)$ is real-valued then $c_{-n} = (c_n)^*$

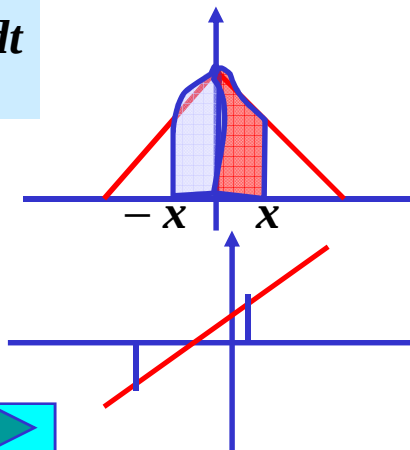
The trigonometric form permits the use of symmetry properties of the function to simplify the computation of coefficients

Even function symmetry

$$f(t) = f(-t)$$

Odd function symmetry

$$f(t) = -f(-t)$$



$$\Rightarrow \int_{-x}^0 f(t) dt = \int_0^x f(t) dt$$

$$\Rightarrow \int_{-x}^0 f(t) dt = - \int_0^x f(t) dt$$

GEAUX

TRIGONOMETRIC SERIES FOR FUNCTIONS WITH EVEN SYMMETRY

$$a_0 = \frac{1}{T_0} \int_{t_1}^{t_1+T_0} f(t) dt$$

$$a_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} f(t) \cos n\omega_0 t dt \quad t_1 = -\frac{T_0}{2}$$

$$b_n = \frac{2}{T_0} \int_{t_1}^{t_1+T_0} f(t) \sin n\omega_0 t dt$$

$$a_n = \frac{4}{T_0} \int_0^{\frac{T_0}{2}} f(t) \cos n\omega_0 t dt, \quad n=1,2,\dots$$

$$a_0 = \frac{2}{T_0} \int_0^{\frac{T_0}{2}} f(t) dt$$

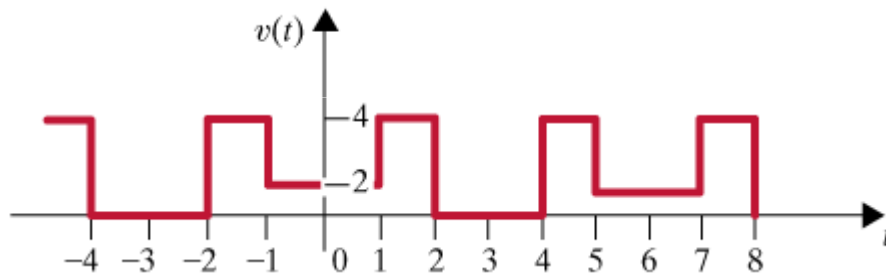
$$2c_n = a_n - jb_n$$

$$2c_{-n} = a_n + jb_n \Rightarrow c_n = c_{-n} = a_n$$

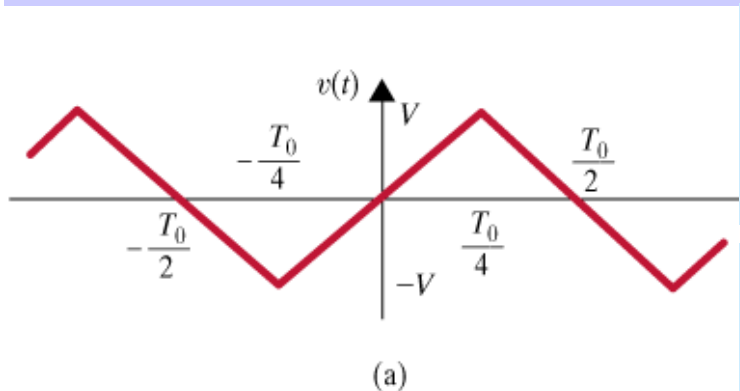
$$b_n = \frac{2}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} f(t) \sin n\omega_0 t dt = 0, \quad n=1,2,\dots$$

$$a_0 = 2$$

$$c_n = n\pi \left[4 \sin \frac{2\pi n}{3} - 2 \sin \frac{\pi n}{3} \right] = a_n, \quad n=1,2,\dots$$



TRIGONOMETRIC SERIES FOR FUNCTIONS WITH ODD SYMMETRY



$$a_n = \frac{2}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} f(t) \cos n\omega_0 t dt = 0, \quad n=0,1,\dots$$

$$c_0 = 0$$

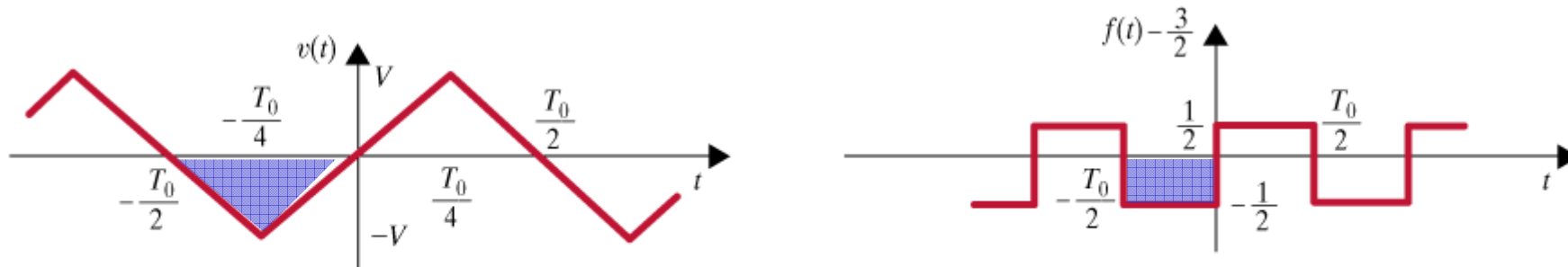
$$b_n = \frac{4}{T_0} \int_0^{\frac{T_0}{2}} f(t) \sin n\omega_0 t dt, \quad n=1,2,\dots$$

$$c_n = c_{-n}^* = -jb_n, \quad n=1,2,\dots$$



FUNCTIONS WITH HALF-WAVE SYMMETRY

$$f(t) = -f\left(t - \frac{T_0}{2}\right), \forall t$$



Examples of signals with half-wave symmetry

Each half cycle is an inverted copy of the adjacent half cycle

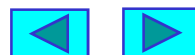
$$a_0 = 0$$

$$a_n = b_n = 0 \quad \text{for } n \text{ even}$$

$$a_n = \frac{4}{T_0} \int_0^{\frac{T_0}{2}} f(t) \cos n\omega_0 t dt \quad \text{for } n \text{ odd}$$

$$b_n = \frac{4}{T_0} \int_0^{\frac{T_0}{2}} f(t) \sin n\omega_0 t dt \quad \text{for } n \text{ odd}$$

There is further simplification if the function is also odd or even symmetric

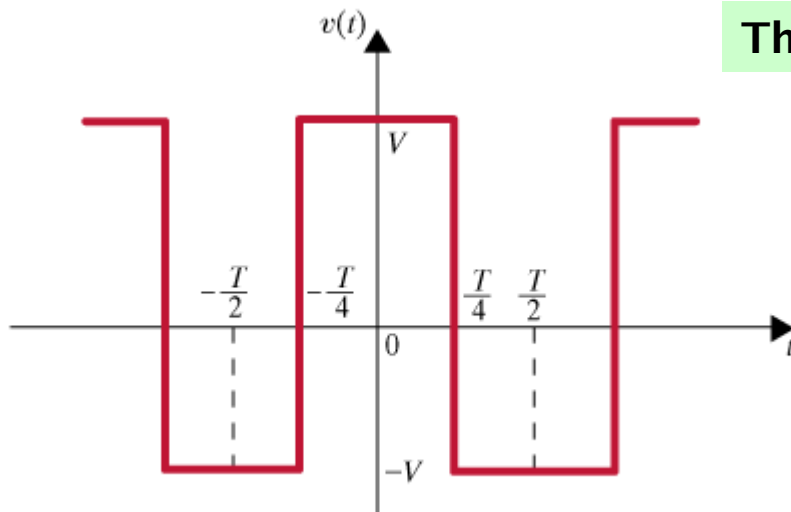


LEARNING EXAMPLE

Find the trigonometric Fourier series coefficients

This is an even function with half-wave symmetry

even $\Rightarrow b_n = 0, n = 1, 2, \dots$

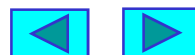


half - wave symmetry $\Rightarrow a_n = \frac{4}{T_0} \int_0^{\frac{T_0}{2}} f(t) \cos n\omega_0 t dt$ for n odd
 $a_n = 0$ for n even

$$\omega_0 = \frac{2\pi}{T} \Rightarrow \omega_0 T = 2\pi$$

$$a_{2k+1} = \frac{4}{T} \left[\int_0^{\frac{T}{4}} V \cos(2k+1)\omega_0 t dt - \int_{\frac{T}{4}}^{\frac{T}{2}} V \cos(2k+1)\omega_0 t dt \right] = \frac{8}{T} \int_0^{\frac{T}{4}} V \cos(2k+1)\omega_0 t dt$$

$$a_{2k+1} = \frac{8}{T(2k+1)\omega_0} \sin(2k+1)\omega_0 \frac{T}{4} = \frac{4V}{(2k+1)\pi} \sin \frac{(2k+1)\pi}{2}$$

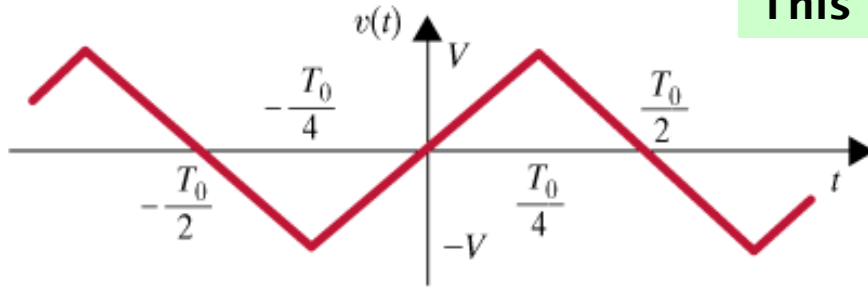


LEARNING EXAMPLE

Find the trigonometric Fourier series coefficients

This is an odd function with half-wave symmetry

odd $\Rightarrow a_n = 0; n = 0, 1, 2, \dots$



half - wave symmetry $\Rightarrow b_n = \frac{4}{T_0} \int_0^{\frac{T_0}{2}} f(t) \sin n \omega_0 t dt$ for n odd
 $b_n = 0$ for n even

$$b_{2k+1} = \frac{4}{T_0} \left[\int_0^{\frac{T_0}{4}} \frac{4V}{T_0} t \sin[(2k+1)\omega_0 t] dt + \int_{\frac{T_0}{4}}^{\frac{T_0}{2}} -\frac{4V}{T_0} (t - \frac{T_0}{2}) \sin[(2k+1)\omega_0 t] dt \right]$$

Use change of variable to show that the two integrals have the same value

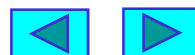
$$b_{2k+1} = \frac{8 \times 4V}{T_0^2} \int_0^{\frac{T_0}{4}} t \sin[(2k+1)\omega_0 t] dt$$

$$\frac{\omega_0 T_0}{4} = \frac{\pi}{2}$$

$$\int t \sin \omega t dt = -\frac{t \cos \omega t}{\omega} + \frac{\sin \omega t}{\omega^2}$$

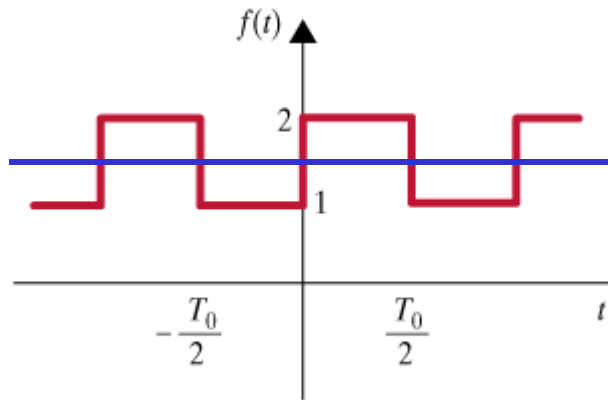
$$b_{2k+1} = \frac{32V}{T_0^2} \left[-\frac{t \cos[(2k+1)\omega_0 t]}{(2k+1)\omega_0} + \frac{\sin[(2k+1)\omega_0 t]}{[(2k+1)\omega_0]^2} \right]_0^{\frac{T_0}{4}}$$

$$b_{2k+1} = \frac{8V}{[(2k+1)\pi]^2} \sin \frac{(2k+1)\pi}{2}$$



LEARNING EXAMPLE

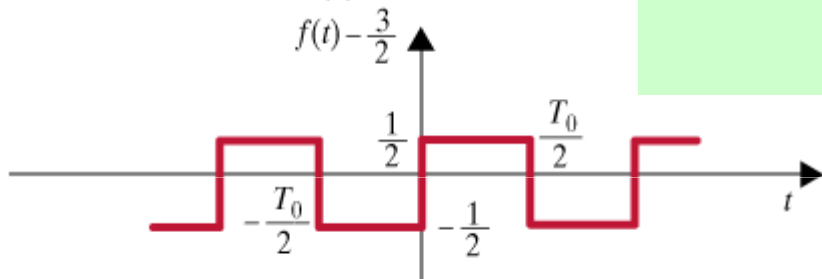
Find the trigonometric Fourier series coefficients



$f(t) - \frac{3}{2}$ is odd with half - wave symmetry

odd $\Rightarrow a_n = 0; n = 0, 1, 2, \dots$

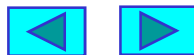
half - wave symmetry $\Rightarrow b_n = \frac{4}{T_0} \int_0^{\frac{T_0}{2}} f(t) \sin n\omega_0 t dt$ for n odd
 $b_n = 0$ for n even



$$b_{2k+1} = \frac{4}{T_0} \int_0^{\frac{T_0}{2}} \frac{1}{2} \sin[(2k+1)\omega_0 t] dt = \frac{2}{T_0(2k+1)\omega_0} [-\cos[(2k+1)\omega_0 t]]_0^{\frac{T_0}{2}}$$

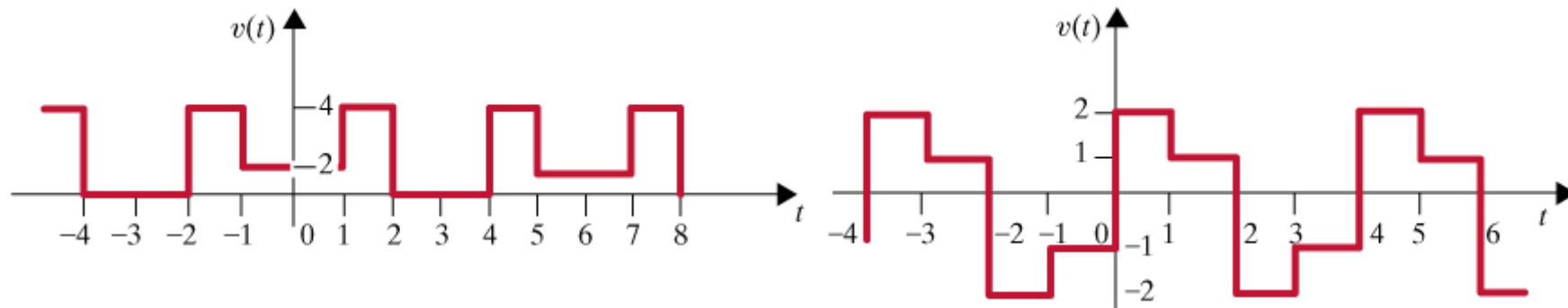
$$b_{2k+1} = \frac{1}{(2k+1)\pi} (1 - \cos[(2k+1)\pi]) = \frac{2}{(2k+1)\pi}$$

$$\therefore f(t) = \frac{3}{2} + \sum_{k=0}^{\infty} \frac{2}{(2k+1)\pi} \sin(2k+1)\omega_0 t$$



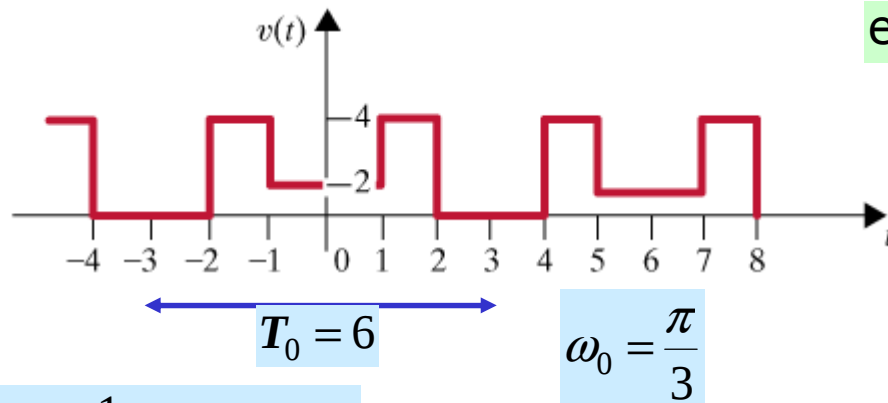
LEARNING EXTENSION

Determine the type of symmetry of the signals



LEARNING EXTENSION

Determine the trigonometric Fourier series expansion



even function $\Rightarrow b_n = 0$

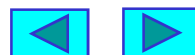
$$a_0 = \frac{2}{T_0} \int_0^{\frac{T_0}{2}} f(t) dt$$

$$a_n = \frac{4}{T_0} \int_0^{\frac{T_0}{2}} f(t) \cos n \omega_0 t dt, \quad n = 1, 2, \dots$$

$$a_0 = \frac{1}{3} \times (2 + 4) = 2$$

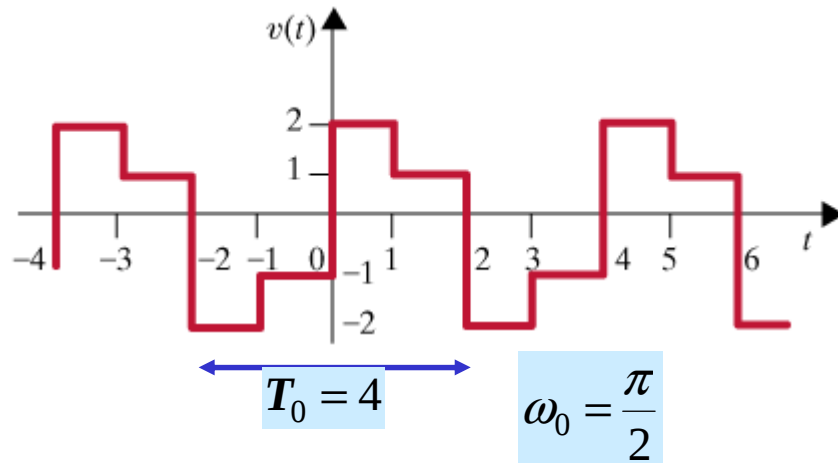
$$a_n = \frac{2}{3} \int_0^1 2 \cos n \omega_0 t dt + \frac{2}{3} \int_1^2 4 \cos n \omega_0 t dt$$

$$a_n = \frac{4}{n\pi} \left(2 \sin \frac{2\pi n}{3} - \sin \frac{n\pi}{3} \right)$$



LEARNING EXTENSION

Determine the trigonometric Fourier series expansion



Half-wave symmetry

$$a_0 = 0$$

$$a_n = b_n = 0 \quad \text{for } n \text{ even}$$

$$a_n = \frac{4}{T_0} \int_0^{\frac{T_0}{2}} f(t) \cos n\omega_0 t dt \quad \text{for } n \text{ odd}$$

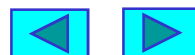
$$b_n = \frac{4}{T_0} \int_0^{\frac{T_0}{2}} f(t) \sin n\omega_0 t dt \quad \text{for } n \text{ odd}$$

$$a_n = \int_0^1 2 \cos \frac{n\pi}{2} t dt + \int_1^2 \cos \frac{n\pi}{2} t dt, \text{ for } n \text{ odd}$$

$$b_n = \int_0^1 2 \sin \frac{n\pi}{2} t dt + \int_1^2 \sin \frac{n\pi}{2} t dt, \text{ for } n \text{ odd}$$

$$a_{2k+1} = \frac{2}{(2k+1)\pi} \sin \frac{(2k+1)\pi}{2}$$

$$b_{2k+1} = \frac{2}{(2k+1)\pi} (2 - \cos(2k+1)\pi)$$

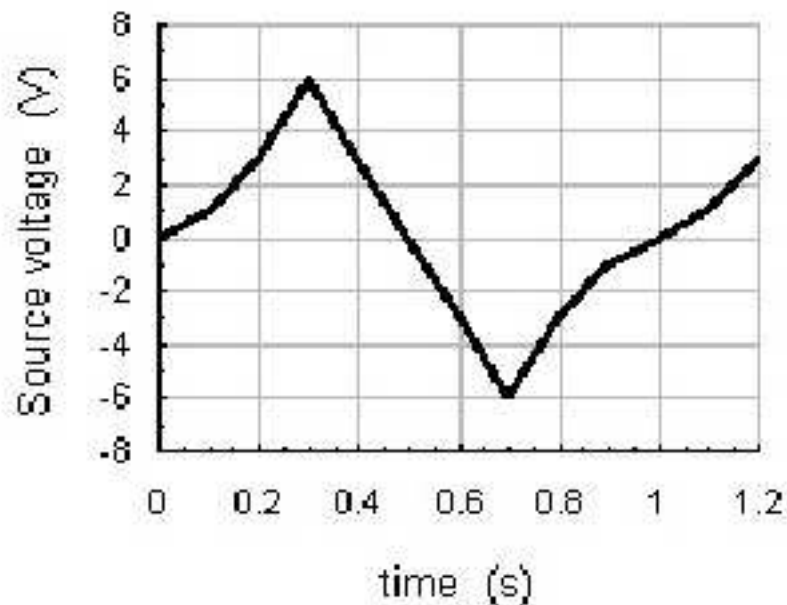


Fourier Series Via PSPICE Simulation

1. Create a suitable PSPICE schematic
2. Create the waveform of interest
3. Set up simulation parameters
4. View the results

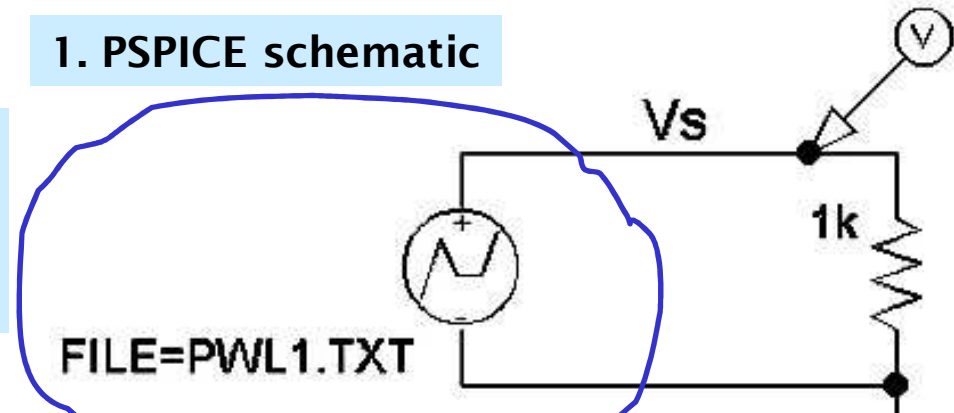
LEARNING EXAMPLE

Determine the Fourier Series of this waveform

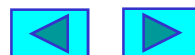
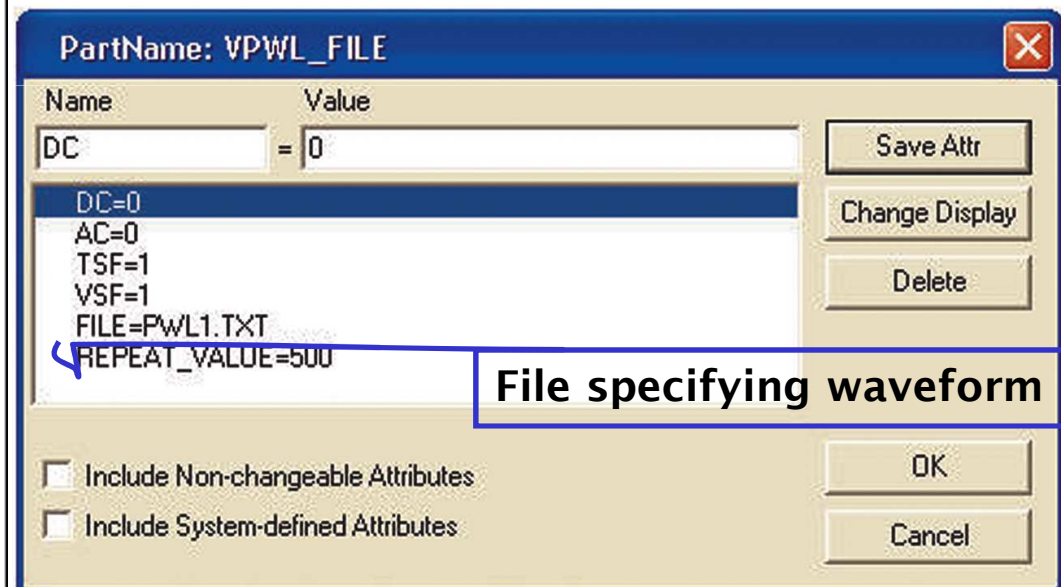


(a)

1. PSPICE schematic



VPWL_FILE in PSPICE library
piecewise linear periodic voltage source

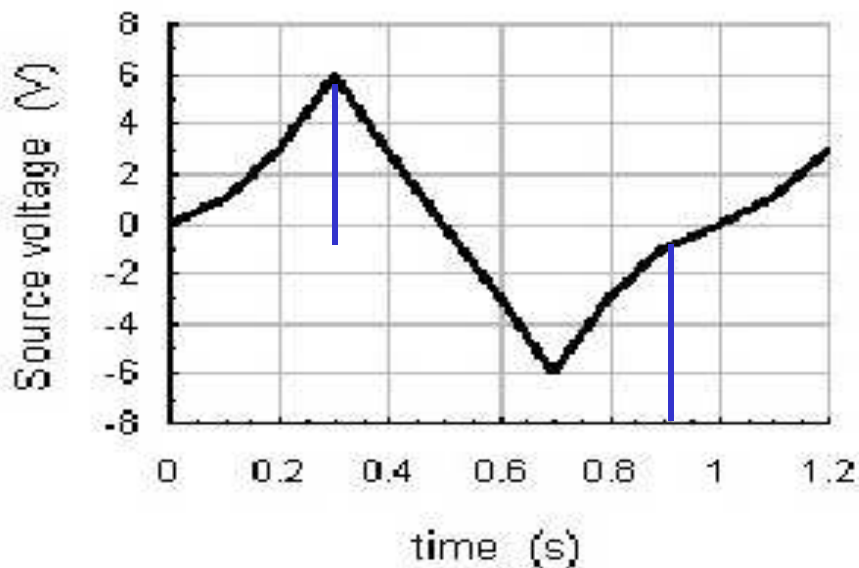


Text file defining corners of piecewise linear waveform

```
* Piecewise Linear File for  
* BECA 7 Fourier Series Example  
*
```

comments

```
0,0  
0.1,1  
0.2,3  
0.3,6  
0.4,3  
0.5,0  
0.6,-3  
0.7,-6  
0.8,-3  
0.9,-1  
1.0,0
```



(a)

Use transient analysis

Transient

Transient Analysis

Print Step: 1
Final Time: 1
No-Print Delay:
Step Ceiling: 1m

☐ Detailed Bias Pt.
☐ Skip initial transient solution

Fourier Analysis

☒ Enable Fourier
Center Frequency: 1
Number of harmonics: 20
Output Vars.: V(Vs)

Fundamental frequency (Hz)

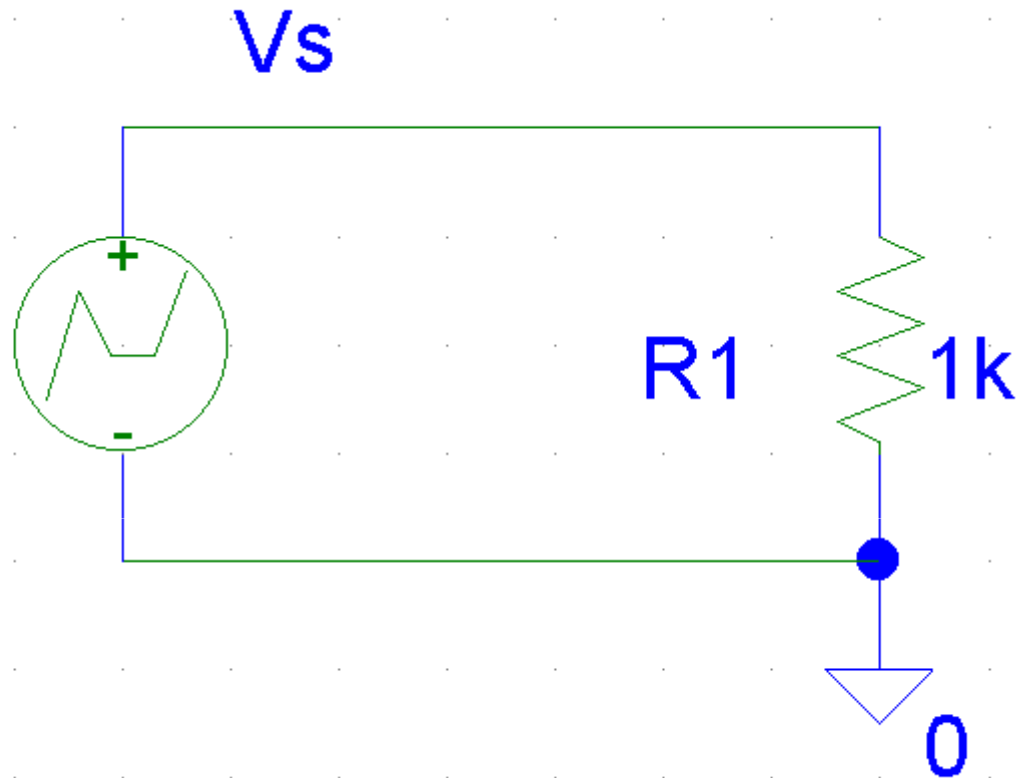
OK

Cancel

GEAUX

Schematic used for Fourier series example

FILE=pwl1.txt



To view result:
From PROBE menu
View/Output File
and search until you find the
Fourier analysis data

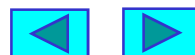
Accuracy of simulation is affected by
setup parameters.

Decreases with number of cycles
increases with number of points



TABLE 15.1 Fourier analysis results for Example 15.5. Fourier components of transient response $V(V_s)$ DC COMPONENT = $-1.353267\text{E-}08$

HARMONIC NO.	FREQUENCY (Hz)	FOURIER COMPONENT	NORMALIZED COMPONENT	PHASE (DEG)	NORMALIZED PHASE (DEG)
1	1.000E+00	4.222E+00	1.000E+00	2.969E-07	0.000E+00
2	2.000E+00	1.283E+00	3.039E-01	1.800E+02	1.800E+02
3	3.000E+00	4.378E-01	1.037E-00	-1.800E+02	-1.800E+02
4	4.000E+00	3.838E-01	9.090E-02	4.620E-07	-7.254E-07
5	5.000E+00	1.079E-04	2.556E-05	1.712E-03	1.711E-03
6	6.000E+00	1.703E-01	4.034E-02	1.800E+02	1.800E+02
7	7.000E+00	8.012E-02	1.898E-02	-9.548E-06	-1.163E-05
8	8.000E+00	8.016E-02	1.899E-02	5.191E-06	2.816E-06
9	9.000E+00	5.144E-02	1.218E-02	-1.800E+02	-1.800E+02
10	1.000E+01	1.397E-04	3.310E-05	1.800E+02	1.800E+02
11	1.100E+01	3.440E-02	8.149E-03	-1.112E-04	-1.145E-04
12	1.200E+01	3.531E-02	8.364E-03	1.800E+02	1.800E+02
13	1.300E+01	2.343E-02	5.549E-03	1.800E+02	1.800E+02
14	1.400E+01	3.068E-02	7.267E-03	-3.545E-05	-3.960E-05
15	1.500E+01	3.379E-04	8.003E-05	-3.208E-03	-3.212E-03
16	1.600E+01	2.355E-02	5.579E-03	-1.800E+02	-1.800E+02
17	1.700E+01	1.309E-02	3.101E-03	2.905E-04	2.854E-04
18	1.800E+01	1.596E-02	3.781E-03	-5.322E-05	-5.856E-05
19	1.900E+01	1.085E-02	2.569E-03	-1.800E+02	-1.800E+02
20	2.000E+01	2.994E-04	7.092E-05	1.800E+02	1.800E+02

Total harmonic distortion = $3.378352\text{E+}01$ percent

TIME-SHIFTING

It is easier to study the effect of time-shift with the exponential series expansion

$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

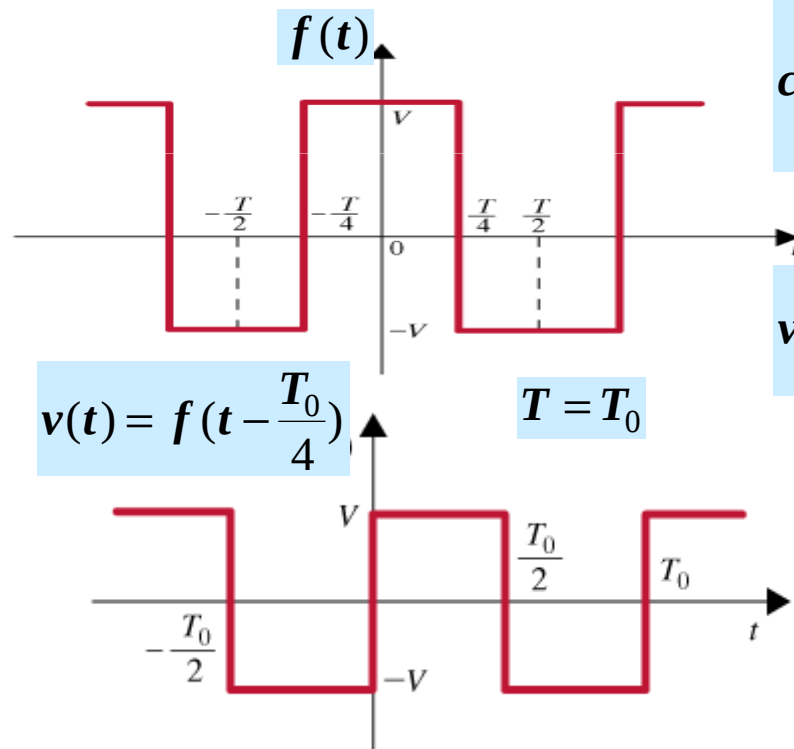
$$f(t-t_0) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0(t-t_0)} = \sum_{n=-\infty}^{\infty} (c_n e^{-jn\omega_0 t_0}) e^{jn\omega_0 t} =$$

Time shifting the function only changes the phase of the coefficients

t_0 = time shift for $f(t)$
 $-n\omega_0 t_0$ = phase shift for c_n

$$\omega_0 T_0 = 2\pi$$

LEARNING EXAMPLE



$$c_n = \begin{cases} 0 & n \text{ even} \\ \frac{2V}{\pi n} \sin \frac{n\pi}{2} & n \text{ odd} \end{cases}$$

$$-n\omega_0 t_0 = -n\omega_0 \frac{T_0}{4} = -\frac{n\pi}{2}$$

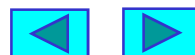
$$v(t) = \sum_{n=-\infty}^{\infty} c_{vn} e^{jn\omega_0 t}$$

$$c_{vn} = \begin{cases} 0 & n \text{ even} \\ \frac{2V}{\pi n} \sin \frac{n\pi}{2} e^{-j\frac{n\pi}{2}} & n \text{ odd} \end{cases}$$

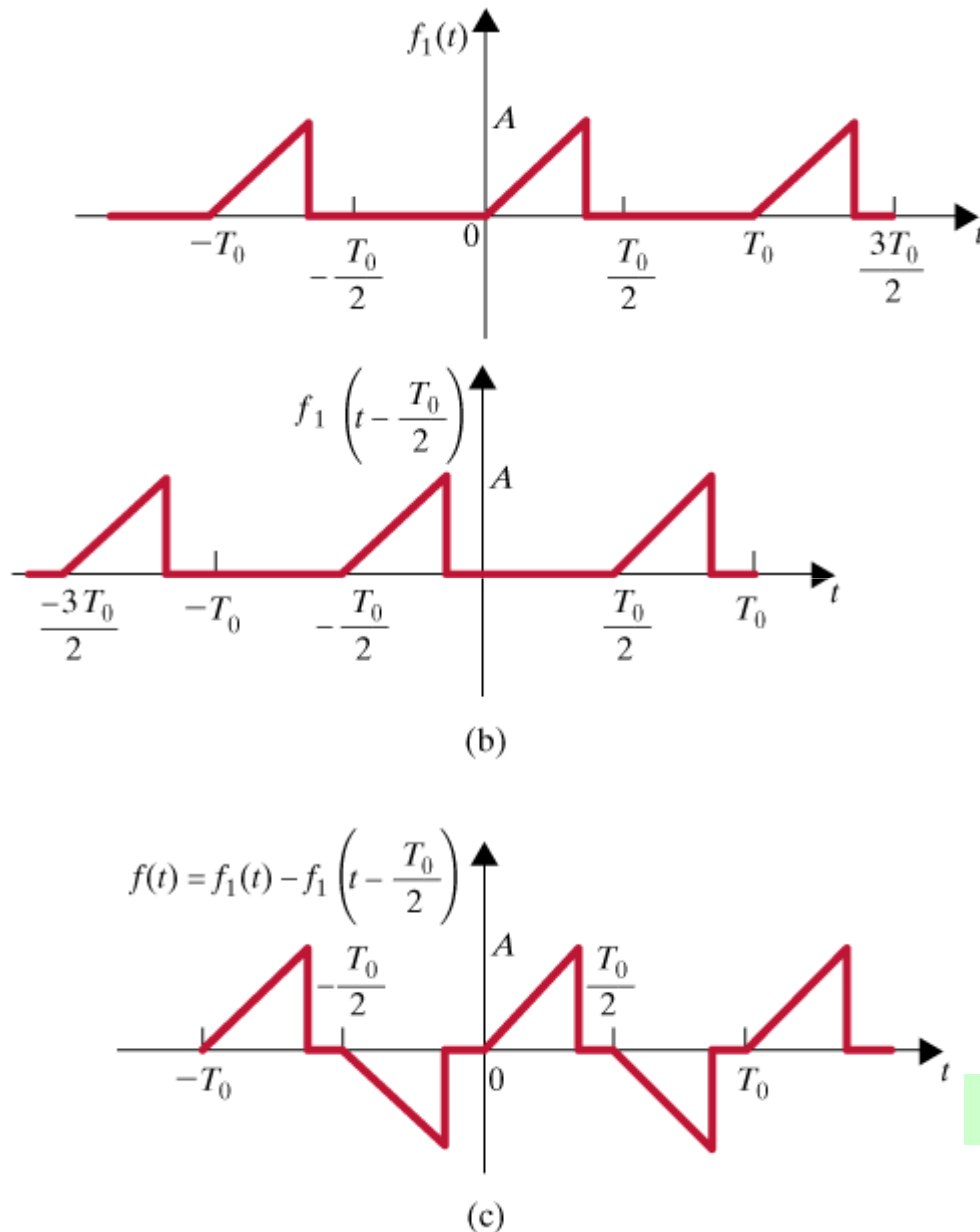
$$e^{-jn\frac{\pi}{2}} = \cos n\frac{\pi}{2} - j \sin n\frac{\pi}{2}$$

$$\left(\sin n\frac{\pi}{2} \right)^2 = 1$$

$$c_{v,2k+1} = -j \frac{2V}{\pi n}$$



Time shifting and half-wave periodic signals



Every half-wave periodic signal, $f(t)$, with period, T_0 , can be expressed as

$$f(t) = f_1(t) - f_1\left(t - \frac{T_0}{2}\right)$$

$$\text{with } f_1(t) = \begin{cases} f(t) & 0 \leq t < \frac{T_0}{2} \\ 0 & \frac{T_0}{2} \leq t < T_0 \end{cases}$$

$$\text{Assume } f_1(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

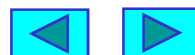
$$\omega_0 T_0 = 2\pi$$

$$f_1\left(t - \frac{T_0}{2}\right) = \sum_{n=-\infty}^{\infty} \left(c_n e^{-jn\omega_0 \frac{T_0}{2}} \right) e^{jn\omega_0 t} =$$

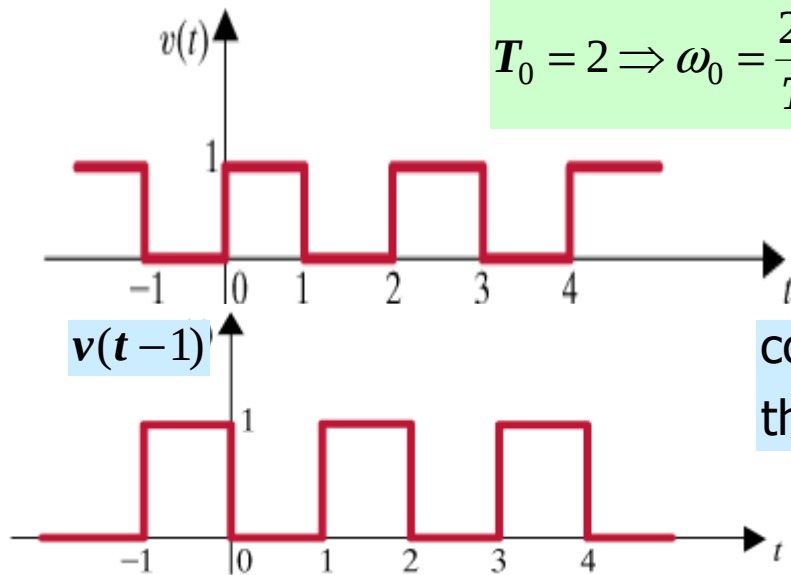
$$e^{-jn\pi} = (-1)^n, \forall n$$

$$\therefore f(t) = \sum_{k=-\infty}^{\infty} 2c_{2k+1} e^{-j(2k+1)\omega_0 t}$$

Only the odd coefficients of f_1 are used



LEARNING EXTENSION



$$T_0 = 2 \Rightarrow \omega_0 = \frac{2\pi}{T_0} = \pi$$

It was shown before
that for $v(t)$

$$c_0 = \frac{1}{2}$$

$$c_n = \frac{1 - e^{-jn\pi}}{j2\pi n}, n \neq 0$$

compute the coefficients for $v(t-1)$ and show that they differ by a πn angle from those of $v(t)$.

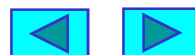
For the delayed function

$$c_{0d} = \frac{1}{2} \int_0^2 v(t-1) dt = \frac{1}{2}$$

For $n \neq 0$

$$c_{nd} = \frac{1}{2} \int_0^2 v(t-1) e^{-jn\pi t} dt = \frac{1}{2} \int_1^3 e^{-jn\pi t} dt = \frac{1}{j2\pi n} [e^{-jn\pi} - e^{-j2n\pi}] = \frac{e^{-jn\pi}}{j2\pi n} [1 - e^{-jn\pi}]$$

$$c_{nd} = e^{-jn\pi} c_n$$



WAVEFORM GENERATION

Assume $f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$

$f(\alpha t) = \sum_{n=-\infty}^{\infty} c_n e^{j\alpha n\omega_0 t}$

Time scaling does not change the values of the series expansion

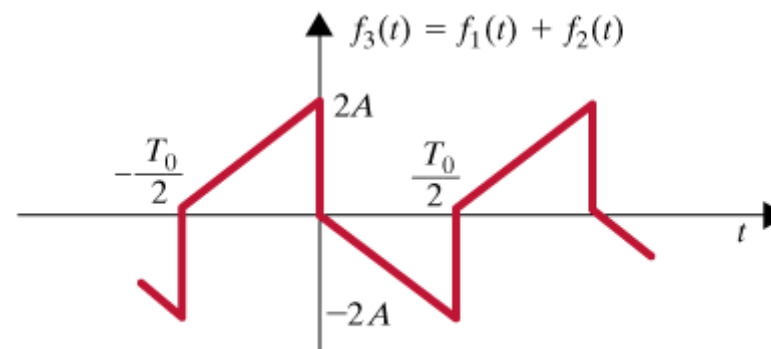
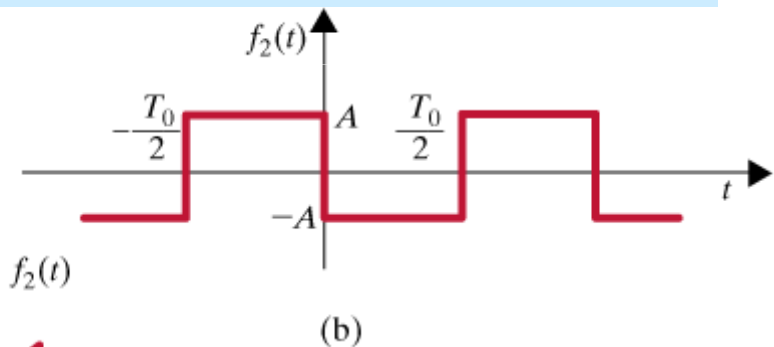
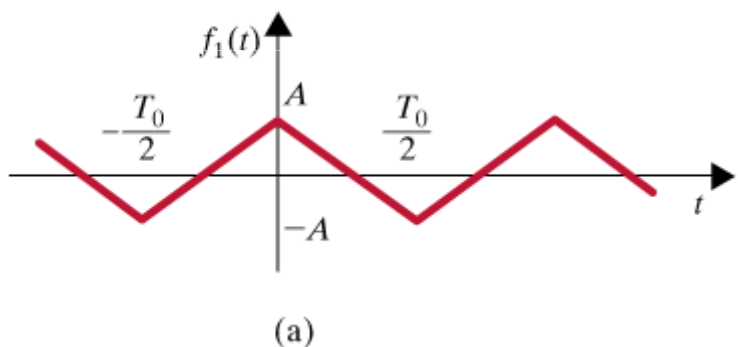
$f(t - t_0) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0(t-t_0)} = \sum_{n=-\infty}^{\infty} (c_n e^{-jn\omega_0 t_0}) e^{jn\omega_0 t}$

Time-shifting modifies the phase of the coefficients

$f(\alpha t - t_0) = \sum_{n=-\infty}^{\infty} (c_n e^{-jn\omega_0 t_0}) e^{j\alpha n\omega_0 t}$

If the Fourier series for $f(t)$ is known then one can easily determine the expansion for any time-shifted and time-scaled version of $f(t)$

The coefficients of a linear combination of signals are the linear combination of the coefficients



$c_{n3} = c_{n1} + c_{n2} \quad \forall n$

One can tabulate the expansions for some basic waveforms and use them to determine the expansions or other signals

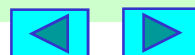
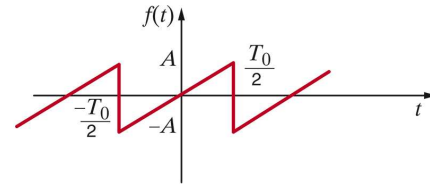
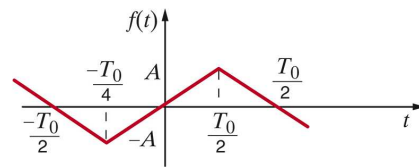


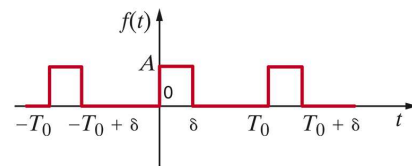
TABLE 15.2 Fourier series for some common waveforms



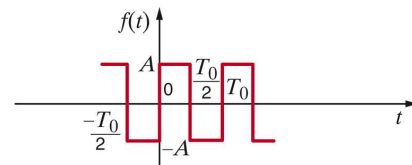
$$f(t) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2A}{n\pi} \sin n\omega_0 t$$



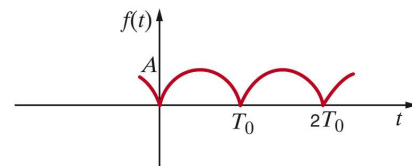
$$f(t) = \sum_{n \text{ odd}} \frac{8A}{n^2 \pi^2} \sin \frac{n\pi}{2} \sin n\omega_0 t$$



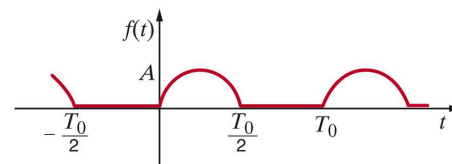
$$f(t) = \sum_{n=-\infty}^{\infty} \frac{A}{n\pi} \sin \frac{n\pi\delta}{T_0} e^{jn\omega_0[t-(\delta/2)]}$$



$$f(t) = \sum_{n=1}^{\infty} \frac{4A}{n\pi} \sin n\omega_0 t$$



$$f(t) = \frac{2A}{\pi} + \sum_{n=1}^{\infty} \frac{4A}{\pi(1-4n^2)} \cos n\omega_0 t$$

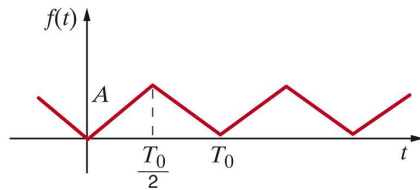


$$f(t) = \frac{A}{\pi} + \frac{A}{2} \sin \omega_0 t + \sum_{n=2}^{\infty} \frac{2A}{\pi(1-n^2)} \cos n\omega_0 t$$

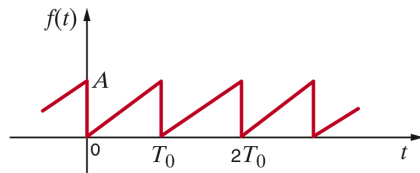
(Continues on the next page)



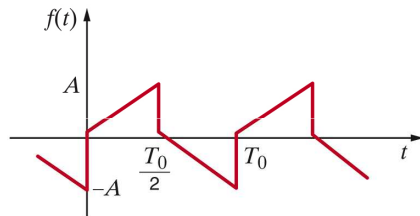
TABLE 15.2 (Continued)



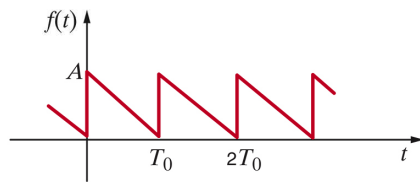
$$f(t) = \frac{A}{2} + \sum_{\substack{n=-\infty \\ n \neq 0 \\ n \text{ odd}}}^{\infty} \frac{-2A}{n^2 \pi^2} e^{jn\omega_0 t}$$



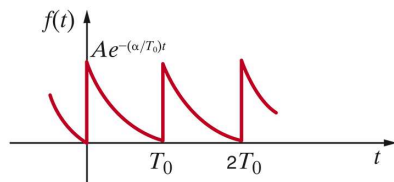
$$f(t) = \frac{A}{2} + \sum_{n=1}^{\infty} \frac{-A}{n\pi} \sin n\omega_0 t$$



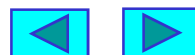
$$f(t) = \sum_{n=1}^{\infty} \frac{-4A}{\pi^2 n^2} \cos n\omega_0 t + \frac{2A}{\pi n} \sin n\omega_0 t$$



$$f(t) = \frac{A}{2} + \sum_{n=1}^{\infty} \frac{A}{\pi n} \sin n\omega_0 t$$

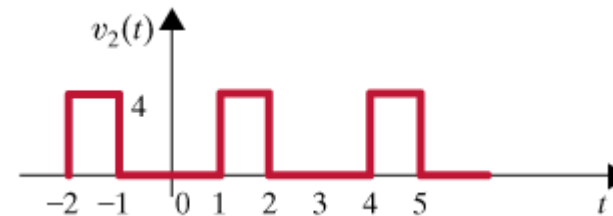


$$f(t) = \sum_{n=-\infty}^{\infty} \frac{A(1 - e^{-\alpha})}{\alpha + j2\pi n} e^{jn\omega_0 t}$$



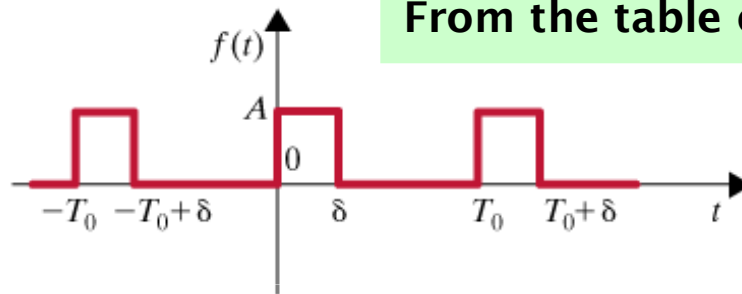
LEARNING EXTENSION

Use the table of Fourier series to determine the expansions of these functions



$$c_0 = \frac{4}{3}$$

From the table of Fourier series



$$f(t) = \sum_{n=-\infty}^{\infty} \frac{A}{n\pi} \sin \frac{n\pi\delta}{T_0} e^{jn\omega_0\left(t - \frac{\delta}{2}\right)}$$

$$c_0 = \frac{A\delta}{T_0}$$

For $v_1(t)$

$$v_1(t) = f(t + \delta/2)$$

$$A = 2, T_0 = 6, \delta = 2$$

$$v_1(t) = \sum_{n=-\infty}^{\infty} \frac{2}{n\pi} \sin \frac{n\pi}{3} e^{jn\frac{\pi}{3}t} \quad c_0 = \frac{2}{3}$$

Strictly speaking the value for $n=0$ must be computed separately.

$$e^{-jn\pi} = (-1)^n$$

For $v_2(t)$

$$v_2(t) = f\left(t - \frac{T_0 - \delta}{2}\right)$$

$$A = 4, T_0 = 3, \delta = 1, \omega_0 = \frac{2\pi}{3}$$

$$v_2(t) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{4}{n\pi} \sin \frac{n\pi}{3} e^{jn\frac{2\pi}{3}\left(t - \frac{3-1}{2} - \frac{1}{2}\right)}$$

$$v_2(t) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{4}{n\pi} \sin \frac{n\pi}{3} e^{-jn\pi} e^{jn\frac{2\pi}{3}t}$$



FREQUENCY SPECTRUM

The spectrum is a graphical display of the coefficients of the Fourier series. The one-sided spectrum is based on the representation

$$f(t) = a_0 + \sum_{n=1}^{\infty} D_n \cos(n\omega_0 t + \theta_n) = a_0 + \sum_{n=1}^{\infty} \operatorname{Re}[D_n \angle \theta_n e^{jn\omega_0 t}]$$

The amplitude spectrum displays D_n as the function of the frequency. The phase spectrum displays the angle θ_n as function of the frequency. The frequency axis is usually drawn in units of fundamental frequency

The two-sided spectrum is based on the exponential representation

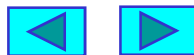
$$f(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

In the two-sided case, the amplitude spectrum plots $|c_n|$ while the phase spectrum plots $\angle c_n$ versus frequency (in units of fundamental frequency)

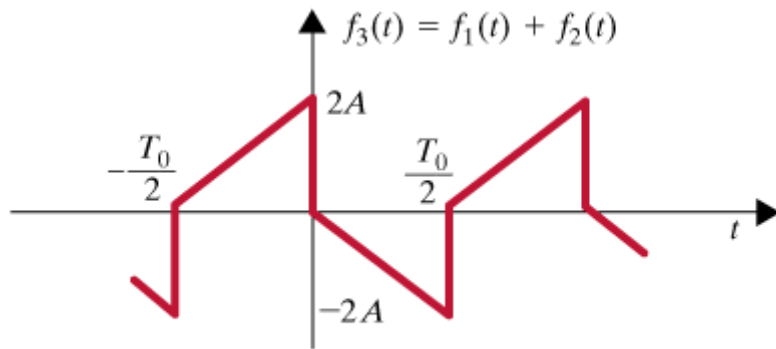
$$D_n \angle \theta_n = 2c_n = a_n - jb_n$$

$$c_{-n} = c_n^*$$

Both spectra display equivalent information



LEARNING EXAMPLE



(c)

The Fourier series expansion, when $A=5$, is given by

$$v(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{20}{n\pi} \sin n\omega_0 t - \frac{40}{n^2\pi^2} \cos n\omega_0 t$$

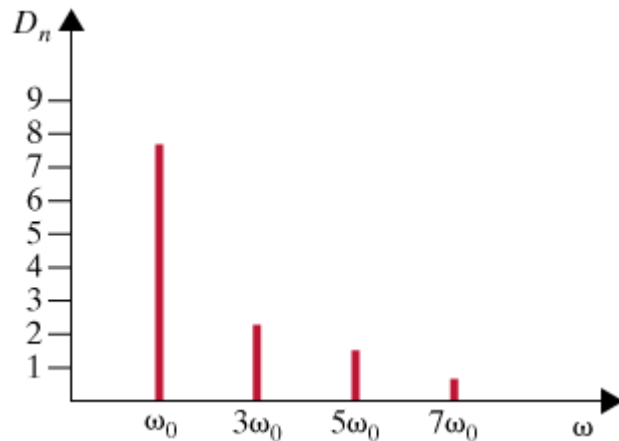
Determine and plot the first four terms of the spectrum

$$D_n \angle \theta_n = 2c_n = a_n - jb_n$$

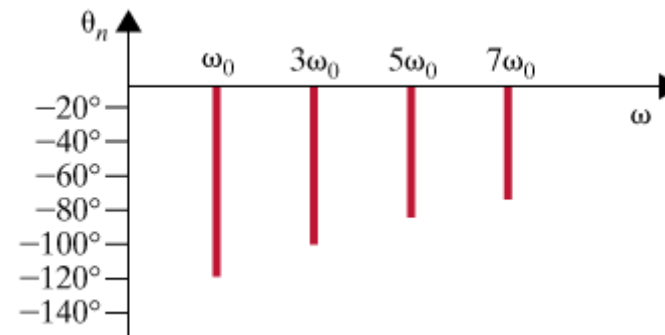
$$D_1 \angle \theta_1 = -\frac{40}{\pi^2} - j\frac{20}{\pi} = 7.5 \angle -122^\circ$$

$$D_3 = -\frac{40}{9\pi^2} - j\frac{20}{3\pi} = 2.2 \angle -102^\circ$$

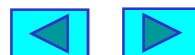
$$D_5 \angle \theta_5 = 1.3 \angle -97^\circ, D_7 \angle \theta_7 = 0.92 \angle -95^\circ$$



Amplitude spectrum

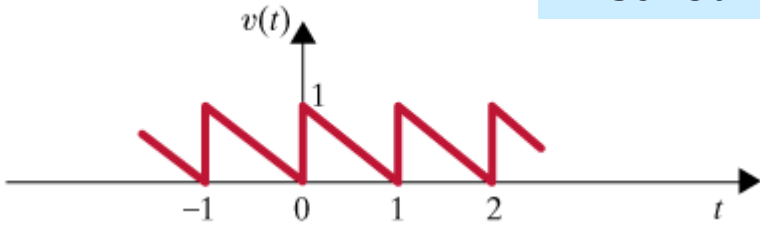


Phase spectrum

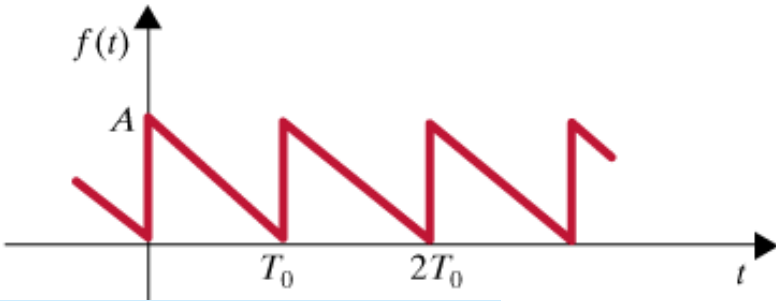


LEARNING EXTENSION

Determine the trigonometric Fourier series and plot the first four terms of the amplitude and phase spectra



$$\mathbf{v}(t) = \frac{1}{2} + \sum_1^{\infty} \frac{1}{\pi n} \sin 2\pi n t$$

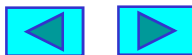
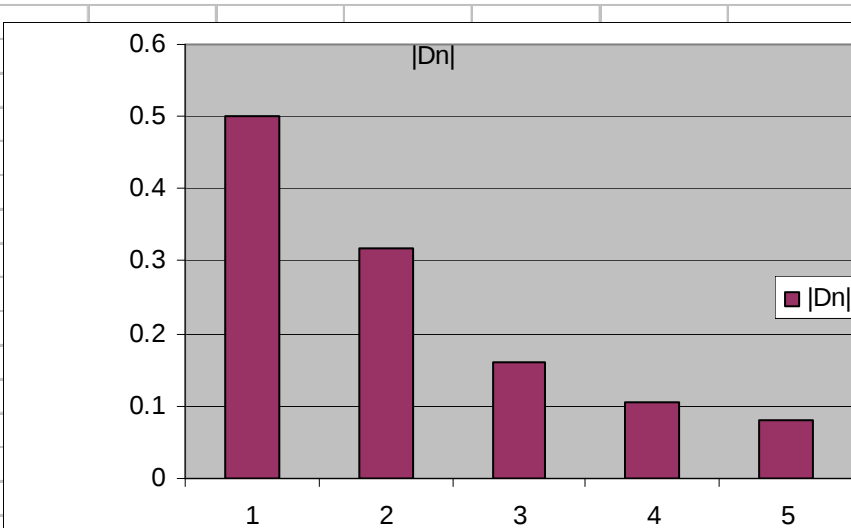


$$f(t) = \frac{A}{2} + \sum_{n=1}^{\infty} \frac{A}{\pi n} \sin n \omega_0 t$$

From the table of series

$$\mathbf{D}_n \angle \theta_n = 2\mathbf{c}_n = \mathbf{a}_n - j\mathbf{b}_n$$

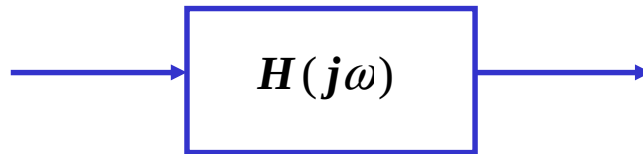
$$D_0 = \frac{1}{2}; D_n = -j \frac{1}{\pi n}; n > 0$$

[illegible]

STEADY STATE NETWORK RESPONSE TO PERIODIC INPUTS

1. Replace the periodic signal by its Fourier series
2. Determine the steady state response to each harmonic
3. Add the steady state harmonic responses

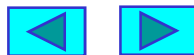
$$H(j\omega) = |H(j\omega)| e^{j\phi(\omega)} = |H(j\omega)| \angle \phi(\omega)^\circ$$



$$|c| e^{j(\omega_1 t + \theta)} \longrightarrow |c| |H(j\omega_1)| e^{j(\omega_1 t + \theta + \phi(\omega_1))}$$

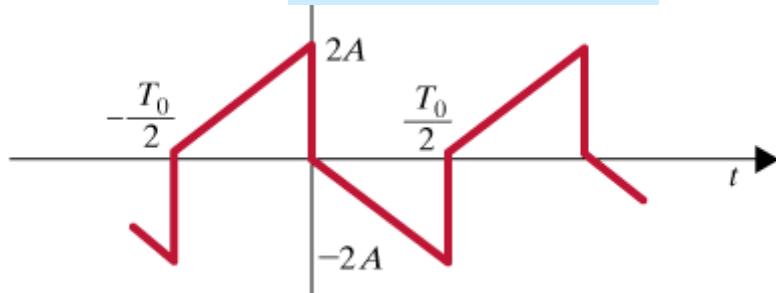
$$|D| \cos(\omega_1 t + \theta) \longrightarrow |D| |H(j\omega_1)| \cos(\omega_1 t + \theta + \phi(\omega_1))$$

$$\text{Re}\{|D| \angle \theta e^{j\omega_1 t}\} \longrightarrow \text{Re}\{|D| \angle \theta |H(j\omega_1)| \angle \phi(\omega_1) e^{j\omega_1 t}\}$$



LEARNING EXAMPLE

$$v(t): A = 5, T_0 = \pi$$



$$v(t) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \left(\frac{20}{n\pi} \sin 2nt - \frac{40}{n^2\pi^2} \cos 2nt \right)$$

$$|4 + j4\omega| = \sqrt{16 + 16\omega^2}$$

$$\angle(4 + j4\omega) = \tan^{-1} \omega$$

$$\omega_n = 2n$$

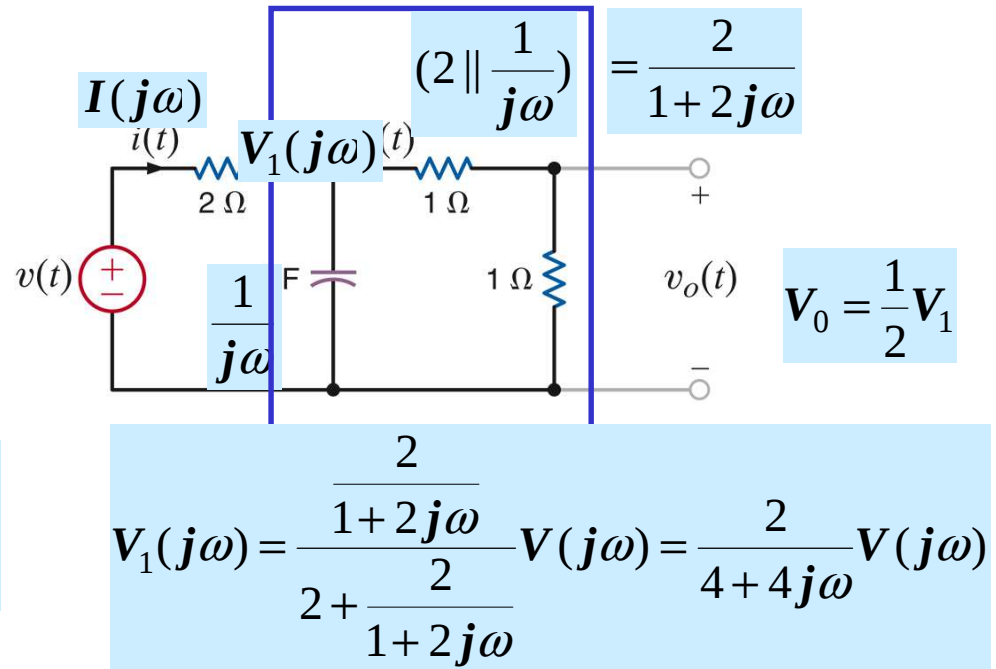
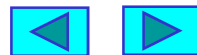
$$H(j\omega) = \frac{1}{4 + j4\omega} = \frac{1}{\sqrt{16 + 16\omega^2}} \angle -\tan^{-1} \omega$$

$$D_n = a_n - jb_n = -\frac{40}{n^2\pi^2} - j\frac{20}{n\pi}$$

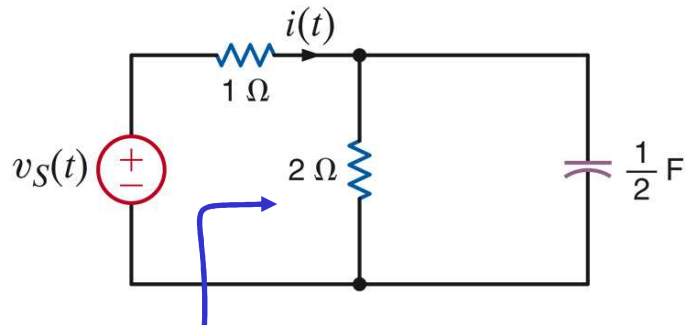
$$|D_n| = \sqrt{\left(\frac{40}{n^2\pi^2}\right)^2 + \left(\frac{20}{n\pi}\right)^2}$$

$$\angle D_n = 180^\circ + \tan^{-1} \frac{n\pi}{2} = -(180^\circ - \tan^{-1} \frac{n\pi}{2})$$

$$v_{on}(t) = \sqrt{\left(\frac{40}{n^2\pi^2}\right)^2 + \left(\frac{20}{n\pi}\right)^2} \frac{1}{\sqrt{16 + 64n^2}} \cos\left(2nt + \tan^{-1} \frac{n\pi}{2} - \tan^{-1} 2n - 180^\circ\right)$$



LEARNING EXTENSION Find the steady state expression for the current $i(t)$



$$Z = 1 + 2 \parallel \frac{2}{j\omega}$$

$$I(j\omega) = \frac{V_S(j\omega)}{Z(j\omega)} = Y(j\omega)V_S(j\omega)$$

$$v_S(t) = \frac{20}{\pi} + \sum_{n=1}^{\infty} \frac{-40}{\pi(4n^2 - 1)} \cos 2nt$$

$$D_0 = \frac{20}{\pi}; D_n = \frac{40}{\pi(4n^2 - 1)} \angle 180^\circ, n \geq 1; \omega_n = 2n$$

$$Z(j\omega) = 1 + \frac{\frac{4}{j\omega}}{2 + \frac{2}{j\omega}} = 1 + \frac{4}{2 + j2\omega} = \frac{6 + j2\omega}{2 + j2\omega} = \frac{3 + j\omega}{1 + j\omega}$$

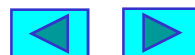
$$I(j2n) = Y(j2n)D_n = \frac{1 + j2n}{3 + j2n} D_n \quad \text{phasor for } n\text{-th harmonic}$$

$$Y(j\omega) = \frac{1 + j\omega}{3 + j\omega}$$

$$Y(0) = \frac{1}{3}$$

$$Y(j2n) = \sqrt{\frac{1 + 4n^2}{9 + 4n^2}} (\angle \tan^{-1} 2n - \angle \tan^{-1} (2n/3))$$

$$i(t) = Y(0)D_0 - \sum_{n=1}^{\infty} |Y_n| |D_n| \cos(2nt - \angle Y_n)$$



AVERAGE POWER

In a network with periodic sources (of the same period) the steady state voltage across any element and the current through are all of the form

$$v(t) = V_{dc} + \sum_{n=1}^{\infty} V_n \cos(n\omega_0 t - \theta_{vn})$$

$$i(t) = I_{dc} + \sum_{n=1}^{\infty} I_n \cos(n\omega_0 t - \theta_{in})$$

The average power is the sum of the average powers for each harmonic

$$\text{Average Power } P = \frac{1}{T} \int_{t_0}^{t_0+T} v(t)i(t)dt$$

$$\int_{t_0}^{t_0+T}$$

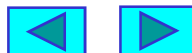
$$v(t)i(t) = V_{dc}I_{dc} + V_{dc} \sum_{n=1}^{\infty} I_n \cos(n\omega_0 t - \theta_{in}) + I_{dc} \sum_{n=1}^{\infty} V_n \cos(n\omega_0 t - \theta_{vn}) + \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} V_{n_1} I_{n_2} \cos(n_1\omega_0 t - \theta_{vn_1}) \cos(n_2\omega_0 t - \theta_{in_2})$$

$$\sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} V_{n_1} I_{n_2} \cos(n_1\omega_0 t - \theta_{vn_1}) \cos(n_2\omega_0 t - \theta_{in_2}) = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \frac{V_{n_1} I_{n_2}}{2} \left[\cos((n_1 + n_2)\omega_0 t - (\theta_{vn_1} + \theta_{in_2})) + \cos((n_1 - n_2)\omega_0 t - (\theta_{vn_1} - \theta_{in_2})) \right]$$

$$\omega_0 = \frac{2\pi}{T} \Rightarrow \int_{t_0}^{t_0+T} \cos(k\omega_0 t + \theta) dt = \begin{cases} 0 & k \neq 0 \\ T & k = 0 \end{cases}$$

$$= T \cos(\theta_{vn} - \theta_{in}) \quad \text{for } n_1 = n_2$$

$$\text{Average Power } P = V_{dc}I_{dc} + \sum_{n=1}^{\infty} \frac{V_n I_n}{2} \cos(\theta_{vn} - \theta_{in})$$



LEARNING EXTENSION**Determine the average power**

$$v(t) = 64 + 36 \cos(377t + 60^\circ) - 24 \cos(754t + 102^\circ) [V]$$

$$i(t) = 1.8 \cos(377t + 45^\circ) + 1.2 \cos(754t + 100^\circ) [A]$$

$$-\cos \alpha = \cos(\alpha - 180^\circ)$$

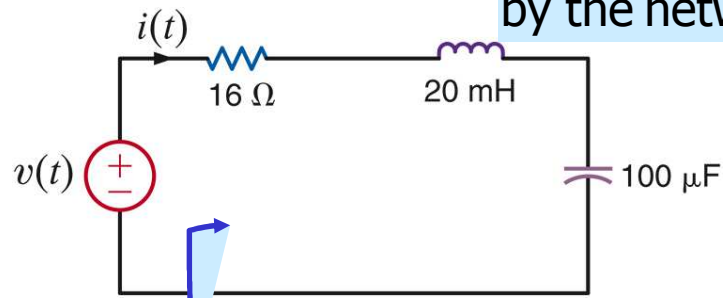
$$P = 0.5(36 \times 1.8 \cos(15^\circ) + 24 \times 1.2 \cos(102^\circ - 180^\circ - 100^\circ))$$

$$P = \frac{62.59 - 28.78}{2} = 16.91 [W]$$



LEARNING EXAMPLE

Determine the current, $i(t)$, and the average power absorbed by the network



$$v(t) = 42 + 16 \cos(377t + 30^\circ) + 12 \cos(754t - 20^\circ) [V]$$

$$Z(j\omega) = R + jL\omega + \frac{1}{jC\omega}$$

$$I(j\omega) = \frac{V(j\omega)}{Z(j\omega)}$$

$$\omega = 377$$

$$V(j377) = 16 \angle 30^\circ,$$

$$Z(j377) = 16 + j0.020 \times 377 - j \frac{1}{10^{-4} \times 377}$$

$$I(j377) = \frac{16 \angle 30^\circ}{16 + j7.54 - j26.53} = 0.64 \angle 79.88^\circ$$

$$\omega = 0$$

capacitor acts as open circuit ($Z = \infty$)

$$I(j0) = 0$$

$$i(t) = 0.64 \sin(377t + 79.88^\circ) + 0.75 \cos(754t - 26.49^\circ)$$

$$\text{Average Power } P = V_{dc} I_{dc} + \sum_{n=1}^{\infty} \frac{V_n I_n}{2} \cos(\theta_{vn} - \theta_{in})$$

$$\omega = 754$$

$$V(j754) = 12 \angle -20^\circ,$$

$$Z(j754) = 16 + j0.020 \times 754 - j \frac{1}{10^{-4} \times 754}$$

$$I(j754) = \frac{12 \angle -20^\circ}{16 + j15.08 - j13.26} = 0.75 \angle -26.49^\circ$$

$$P = 42(0) + \frac{16 \times 0.64 \cos(49.88^\circ) + 12 \times 0.75 \cos(6.49^\circ)}{2}$$



FOURIER TRANSFORM

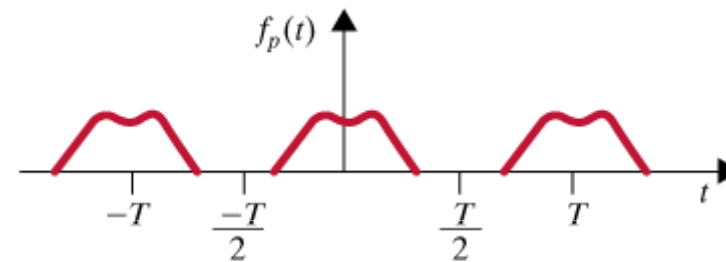
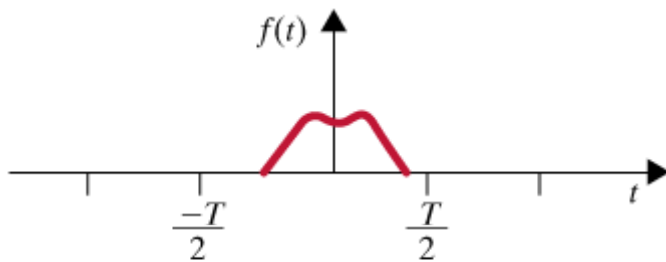
$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$F(\omega) = \mathcal{F}[f(t)]$$

$$f(t) = \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \frac{d\omega}{2\pi}$$

$$f(t) = \mathcal{F}^{-1}[F(\omega)]$$

A heuristic view of the Fourier transform



A non-periodic function can be viewed as the limit of a periodic function when the period approaches infinity

$$f_p(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\frac{2\pi}{T}t}$$

$$f_p(t) = \sum_{n=-\infty}^{\infty} c_n T e^{jn\frac{2\pi}{T}t} \frac{1}{T}$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-jn\frac{2\pi}{T}t} dt$$

$$c_n T = \int_{-T/2}^{T/2} f(t) e^{-jn\frac{2\pi}{T}t} dt$$

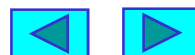
$$\frac{2\pi}{T} = \Delta\omega$$

$$n\Delta\omega \approx \omega_n$$

$$T \rightarrow \infty$$

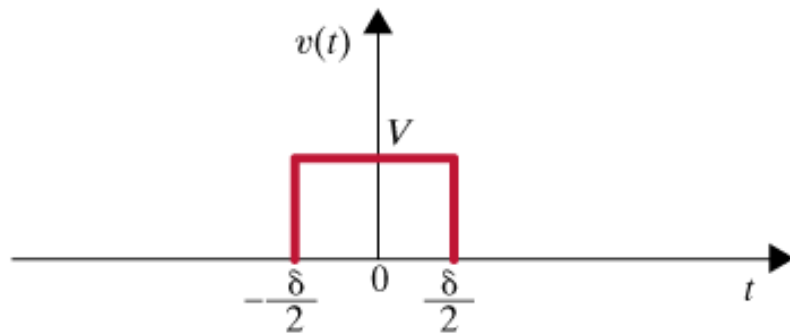
$$\int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \frac{d\omega}{2\pi}$$

$$\int_{-\infty}^{\infty} f(t) e^{j\omega t} dt$$



LEARNING EXAMPLE

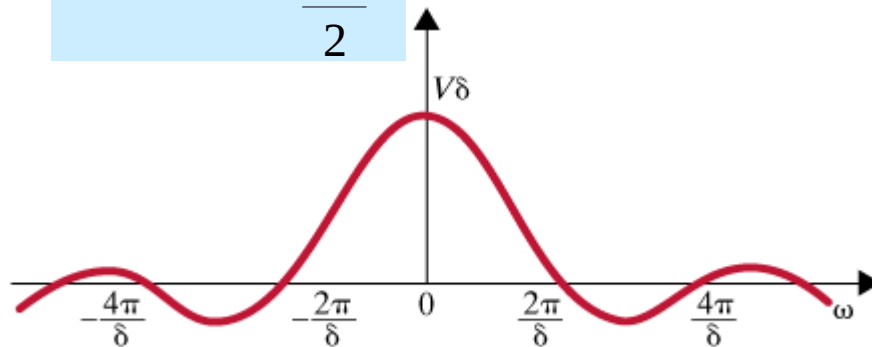
Determine the Fourier transform



$$V(\omega) = \int_{-\infty}^{\infty} v(t) e^{-j\omega t} dt = V \int_{-\delta/2}^{\delta/2} e^{-j\omega t} dt$$

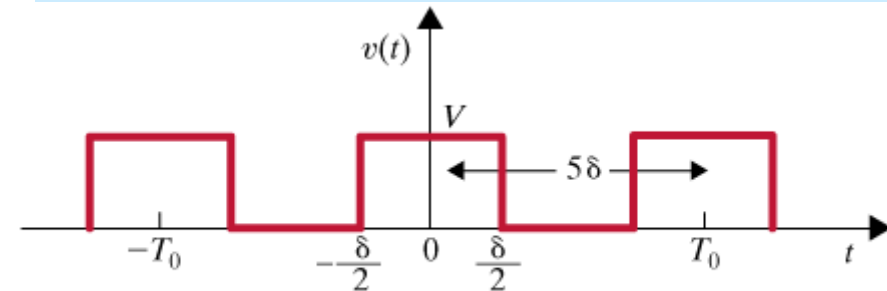
$$V(\omega) = V \left[-\frac{1}{j\omega} e^{-j\omega t} \right]_{-\delta/2}^{\delta/2} = V \frac{e^{j\omega\delta/2} - e^{-j\omega\delta/2}}{j\omega}$$

$$V(\omega) = V\delta \frac{\sin \frac{\omega\delta}{2}}{\frac{\omega\delta}{2}}$$

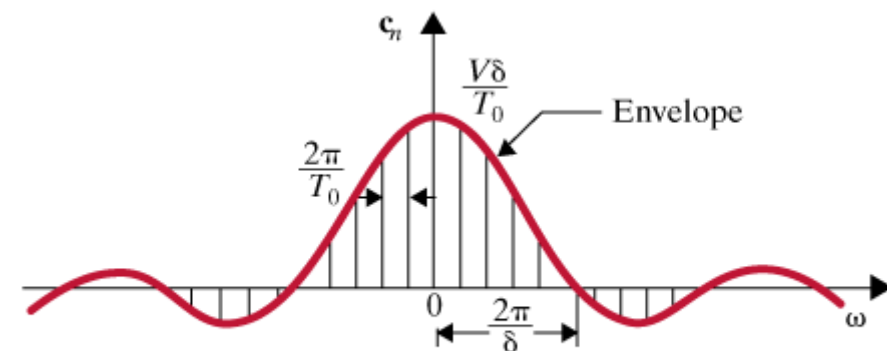


(c)

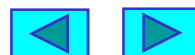
For comparison we show the spectrum of a related periodic function



$$c_n = \frac{V\delta}{T_0} \frac{\sin \frac{\omega\delta}{2}}{\frac{\omega\delta}{2}}$$



Spectrum for $T_0 = 5\delta$



LEARNING EXAMPLES

Determine the Fourier transform of the unit impulse function

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$F(\omega) = \mathcal{F}[f(t)]$$

Sifting or sampling property of the impulse

$$\int_{-\infty}^{\infty} \delta(t-a) f(t) dt = f(a)$$

$$\mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t-a) e^{-j\omega t} dt = e^{-j\omega a}$$

LEARNING EXTENSION

Determine $F(\omega) = \mathcal{F}[\sin \omega_0 t]$

$$f(t) = \sin \omega_0 t = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} \Rightarrow F(\omega) = \frac{2\pi\delta(\omega - \omega_0) - 2\pi\delta(\omega + \omega_0)}{2j}$$

$$F(\omega) = j\pi[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)]$$

Determine the Fourier transform of

$$f(t) = e^{j\omega_0 t}$$

$$f(t) = \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \frac{d\omega}{2\pi}$$

Consider $F(\omega) = 2\pi\delta(\omega - \omega_0)$ and find the corresponding function of time

$$f(t) = \int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0) e^{j\omega t} \frac{d\omega}{2\pi}$$

$$f(t) = e^{j\omega_0 t}$$

$$F(\omega) = \mathcal{F}[e^{j\omega_0 t}] = 2\pi\delta(\omega - \omega_0)$$

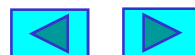


TABLE 15.3 Fourier transform pairs

$f(t)$	$F(\omega)$
$\delta(t - a)$	$e^{-j\omega a}$
A	$2\pi A\delta(\omega)$
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$
$\cos \omega_0 t$	$\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$
$\sin \omega_0 t$	$j\pi\delta(\omega + \omega_0) - j\pi\delta(\omega - \omega_0)$
$e^{-at}u(t), a > 0$	$\frac{1}{a + j\omega}$
$e^{-a t }, a > 0$	$\frac{2a}{a^2 + \omega^2}$
$e^{-at} \cos \omega_0 t u(t), a > 0$	$\frac{j\omega + a}{(j\omega + a)^2 + \omega_0^2}$
$e^{-at} \sin \omega_0 t u(t), a > 0$	$\frac{\omega_0}{(j\omega + a)^2 + \omega_0^2}$

TABLE 15.4 Properties of the Fourier transform

$f(t)$	$F(\omega)$ PROPERTY	
$Af(t)$	$AF(\omega)$	Linearity
$f_1(t) \pm f_2(t)$	$F_1(\omega) \pm F_2(\omega)$	
$f(at)$	$\frac{1}{a} F\left(\frac{\omega}{a}\right), a > 0$	Time-scaling
$f(t - t_0)$	$e^{-j\omega t_0} F(\omega)$	Time-shifting
$e^{j\omega t_0} f(t)$	$F(\omega - \omega_0)$	Modulation
$\frac{d^n f(t)}{dt^n}$	$(j\omega)^n F(\omega)$	Differentiation
$t^n f(t)$	$(j)^n \frac{d^n F(\omega)}{d\omega^n}$	
$\int_{-\infty}^{\infty} f_1(x) f_2(t - x) dx$	$F_1(\omega) F_2(\omega)$	Convolution
$f_1(t) f_2(t)$	$\frac{1}{2\pi} \int_{-\infty}^{\infty} F_1(x) F_2(\omega - x) dx$	



Proof of the convolution property

$$f(t) = f_1(t) \otimes f_2(t) = \int_{-\infty}^{\infty} f_1(x) f_2(t-x) dx$$

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt \\ &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f_1(x) f_2(t-x) dx \right] e^{-j\omega t} dt \end{aligned}$$

Exchanging orders of integration

$$F(\omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f_1(x) f_2(t-x) e^{-j\omega t} dt \right] dx$$

Change integration variable

$$u = t - x \Rightarrow t = u + x$$

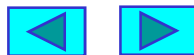
And limits of integration remain the same

$$F(\omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f_1(x) f_2(u) e^{-j\omega(u+x)} du \right] dx$$

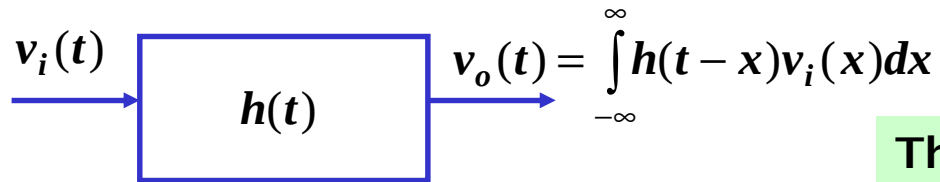
$$F(\omega) = \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f_1(x) f_2(u) e^{-j\omega u} e^{-j\omega x} du \right] dx$$

$$F(\omega) = \int_{-\infty}^{\infty} f_1(x) e^{-j\omega x} dx \left[\int_{-\infty}^{\infty} f_2(u) e^{-j\omega u} du \right]$$

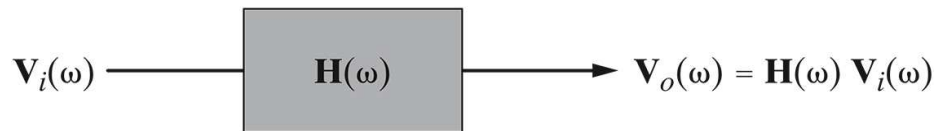
$$F(\omega) = F_1(\omega) F_2(\omega)$$



A Systems application of the convolution property

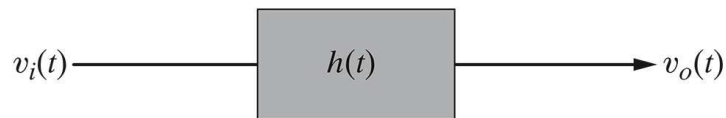


The output (response) of a network can be computed using the Fourier transform



LEARNING EXTENSION

Use Fourier transform to determine $v_o(t)$



$$v_i(t) = e^{-t}u(t)$$

$$h(t) = e^{-2t}u(t)$$

(And all initial conditions are zero)

$$V_o(\omega) = H(\omega)V_i(\omega)$$

$$= \frac{1}{j\omega + 2} \frac{1}{j\omega + 1}$$

$$\frac{1}{(s+2)(s+1)} = \frac{A_1}{s+2} + \frac{A_2}{s+1}$$

Use partial fraction expansion!

From the table of transforms

$$e^{-at}u(t), a > 0$$

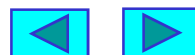
$$\frac{1}{a + j\omega}$$

$$A_1 = (s+2)V_o(s)|_{s=-2} = -1$$

$$A_2 = (s+1)V_o(s)|_{s=-1} = 1$$

$$V_o(\omega) = \frac{1}{j\omega + 1} - \frac{1}{j\omega + 2}$$

$$v_o(t) = (e^{-t} - e^{-2t})u(t)$$



PARSEVAL'S THEOREM

Think of $f(t)$ as a voltage applied to a one Ohm resistor

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |F(\omega)|^2 \frac{d\omega}{2\pi}$$

$$p(t) = v(t)i(t) = |f(t)|^2$$

By definition, the left hand side is the energy of the signal

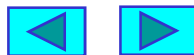
$|f(t)|^2$ = power (or energy density in time)

$|F(\omega)|^2$ = Energy density in the frequency domain

Parseval's theorem permits the determination of the energy of a signal in a given frequency range

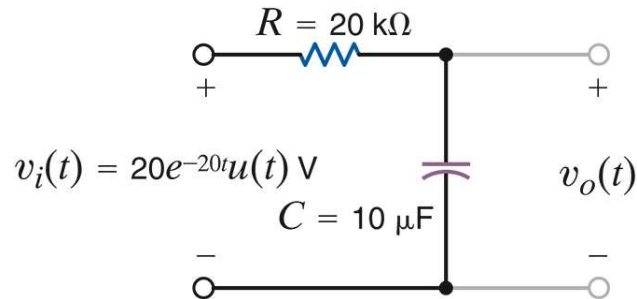
Intuitively, if the Fourier transform has a large magnitude over a frequency range then the signal has significant energy over that range

And if the magnitude of the Fourier transform is zero (or very small) then the signal has no significant energy in that range



LEARNING BY APPLICATION

Examine the effect of this low-pass filter in the quality of the input signal

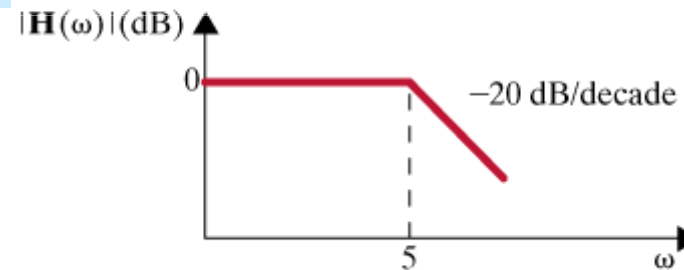
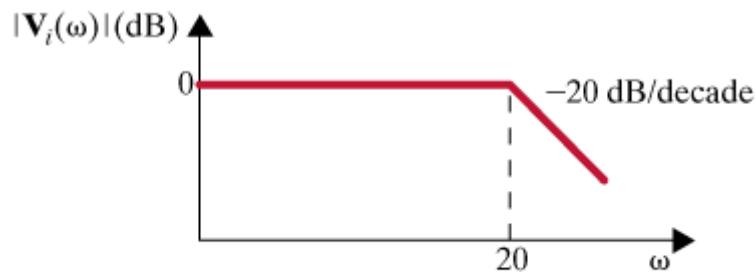


$$V_o(\omega) = H(\omega)V_i(\omega)$$

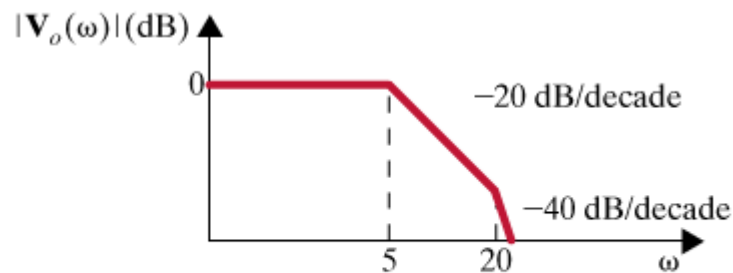
$$V_i(\omega) = \frac{20}{j\omega + 20}$$

$$H(\omega) = \frac{\frac{1}{j\omega C}}{\frac{1}{j\omega C} + R} = \frac{1}{j\omega RC + 1}$$

One can use Bode plots to visualize the effect of the filter



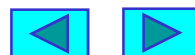
(c)

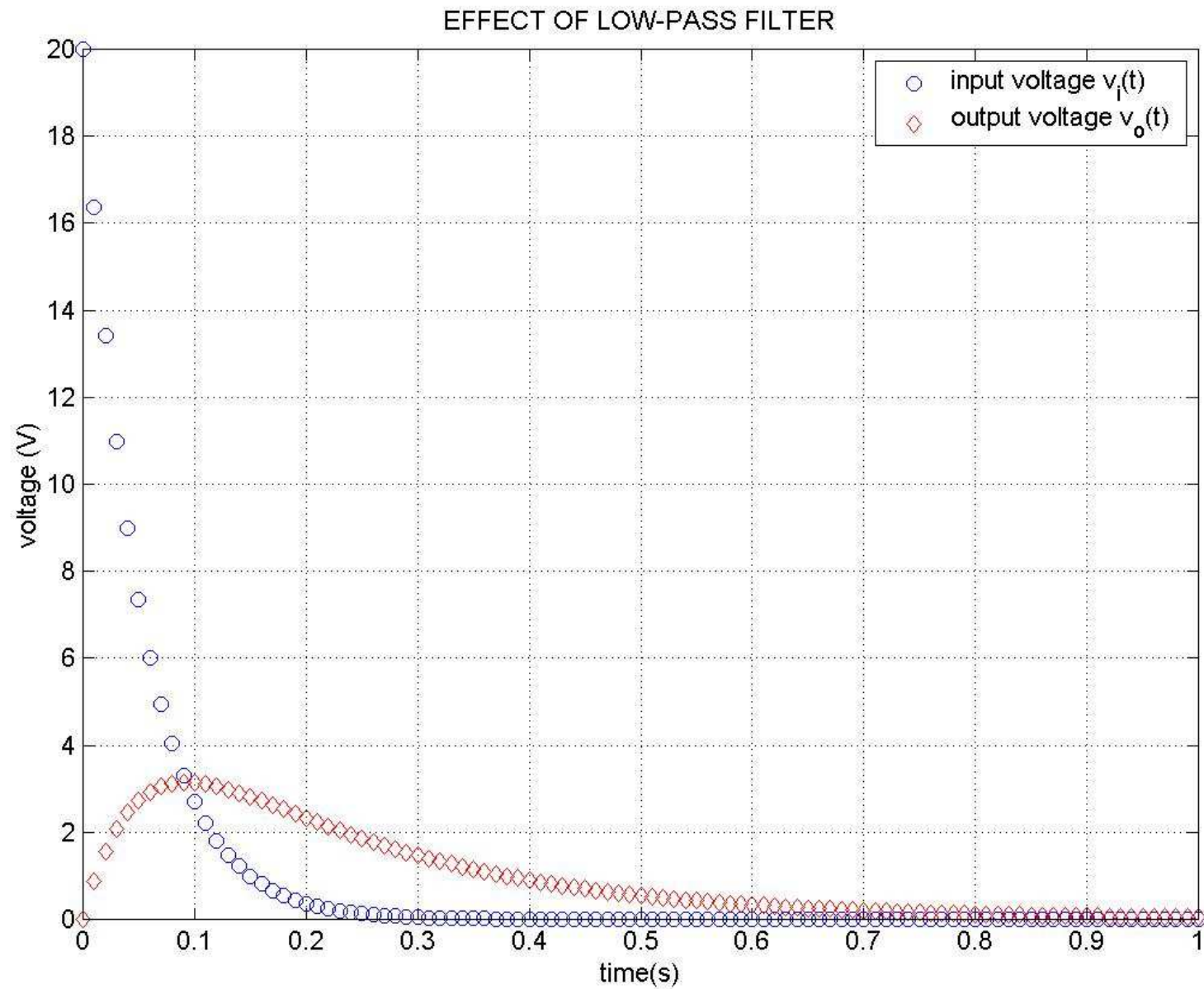


(d)

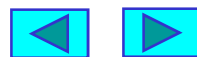
High frequencies in the input signal are attenuated in the output

The effect is clearly visible in the time domain

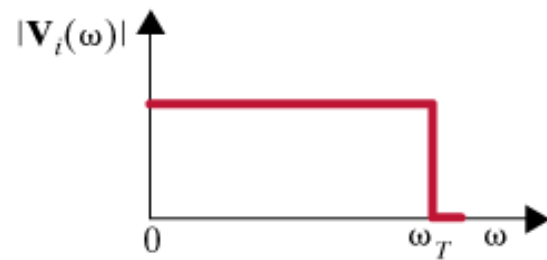




The output signal is slower and with less energy than the input signal

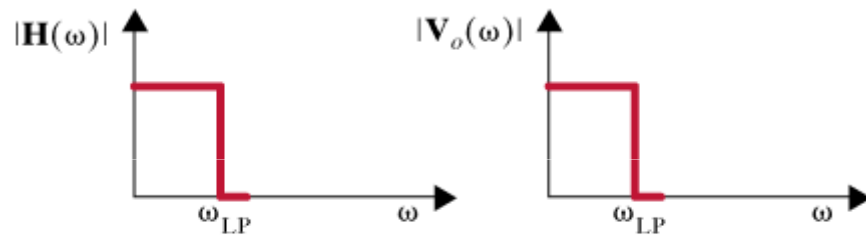


EFFECT OF IDEAL FILTERS



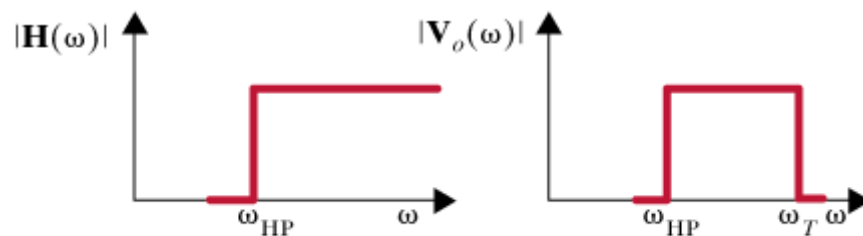
(a)

Effect of low-pass filter



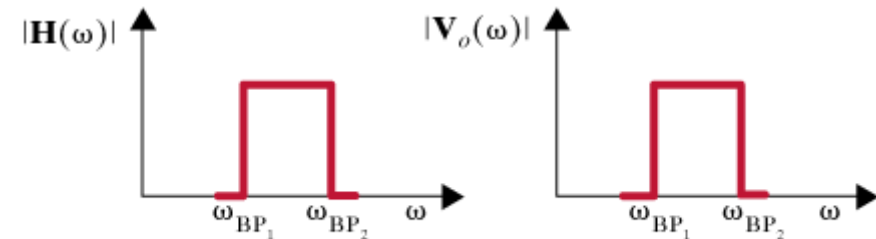
(b)

Effect of high-pass filter



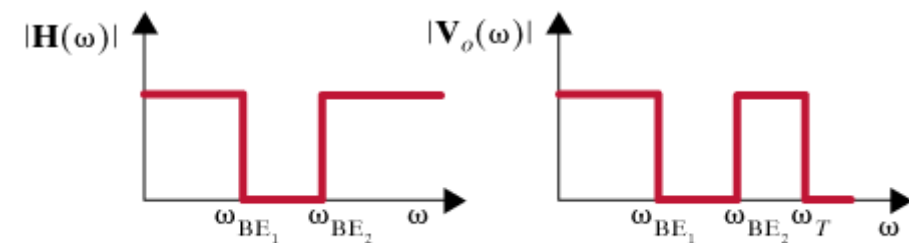
(c)

Effect of band-pass filter



(d)

Effect of band-stop filter



(e)

EXAMPLE AMPLITUDE MODULATED (AM) BROADCASTING

Audio signals do not propagate well in atmosphere - they get attenuated very quickly

Original Solution: Move the audio signals to a different frequency range for broadcast

The frequency range 540kHz - 1700kHz is reserved for AM modulated broadcasting

$$v(t) = [A + s(t)] \cos(\omega_c t) \quad \text{Carrier signals}$$

AM receivers pick a faint copy of $v(t)$

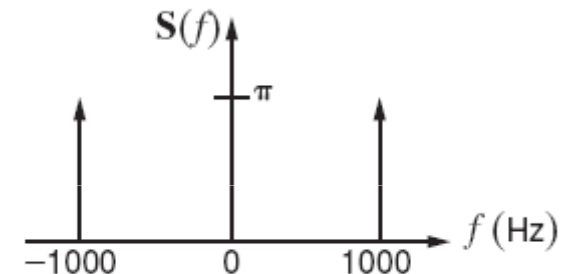
Broadcasted
signal

Audio signal

$$s(t) = \cos(2\pi f_a t) \quad f_a = 1000\text{kHz}; \quad f_c = 900\text{kHz}$$

$$v(t) = (1 + s(t)) \cos 2\pi f_c t$$

$$S(\omega) = \mathcal{F}[\cos(\omega_a t)] = \pi\delta(\omega - \omega_a) + \pi\delta(\omega + \omega_a)$$



$$v(t) = [1 + s(t)] \cos(\omega_c t) = \cos(\omega_c t) + s(t) \cos(\omega_c t)$$

$$\mathcal{F}[\cos(\omega_c t)] = \pi\delta(\omega - \omega_c) + \pi\delta(\omega + \omega_c)$$

$$s(t) \cos(\omega_c t) = s(t) \left\{ \frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2} \right\} \quad \mathcal{F}[s(t) \cos(\omega_c t)] = \frac{1}{2} \{S(\omega - \omega_c) + S(\omega + \omega_c)\}$$

$$\mathcal{F}[s(t) \cos(\omega_c t)] = \mathcal{F}[\cos(\omega_a t) \cos(\omega_c t)]$$

Nothing in audio range!

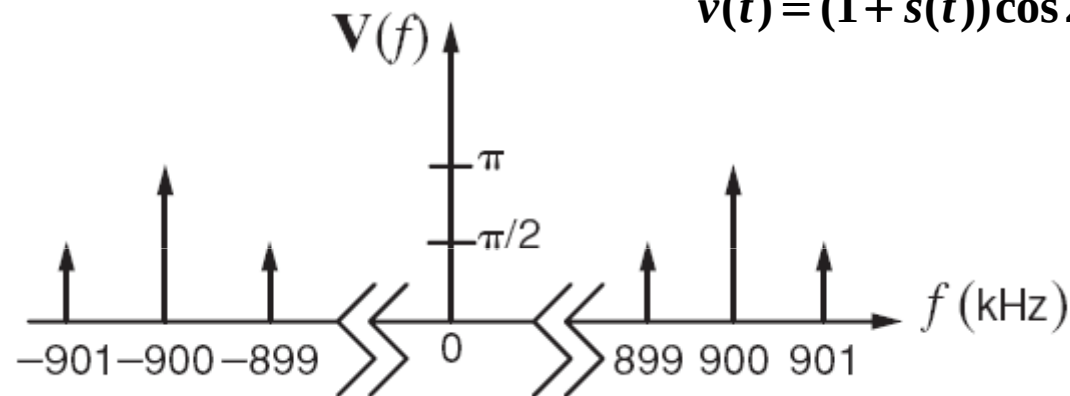
$$\begin{aligned} &= \frac{\pi}{2} \{ \delta(\omega - \omega_a - \omega_c) + \delta(\omega + \omega_a - \omega_c) \\ &\quad + \delta(\omega - \omega_a + \omega_c) + \delta(\omega + \omega_a + \omega_c) \} \end{aligned}$$



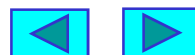
$$V(\omega) = \frac{\pi}{2} \{ 2\delta(\omega - \omega_c) + 2\delta(\omega + \omega_c) + \delta(\omega - \omega_a - \omega_c) \\ + \delta(\omega + \omega_a - \omega_c) + \delta(\omega - \omega_a + \omega_c) + \delta(\omega + \omega_a + \omega_c) \}$$

$$s(t) = \cos(2\pi f_a t) \quad f_a = 1000\text{kHz}; \quad f_c = 900\text{kHz}$$

$$v(t) = (1 + s(t))\cos 2\pi f_c t$$



Audio signal has been AM modulated to the radio frequency range



EXAMPLE

HARMONICS IN POWER SYSTEMS

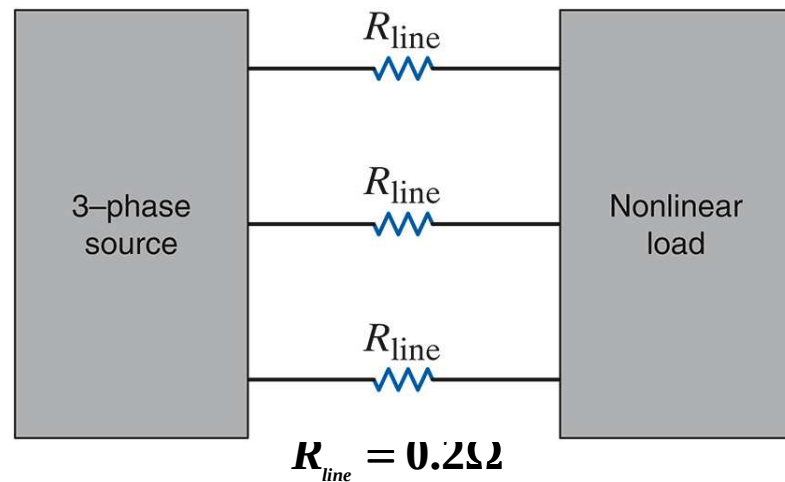


TABLE 15.5 Harmonic line current content for a nonlinear load

HARMONIC	MAGNITUDE (A)
Fundamental	100
1st	5
2nd	30
3rd	1
4th	10

what is the total power loss in the system?

how much of the power loss is caused by the harmonic content?

$$P_{line} = \frac{I_{line}^2}{2} R_{line}$$

TABLE 15.6 Line power loss at the fundamental and each harmonic frequency

HARMONIC	P _{LINE} (W)
Fundamental	1000
1st	0.5
2nd	90
3rd	0.1
4th	10

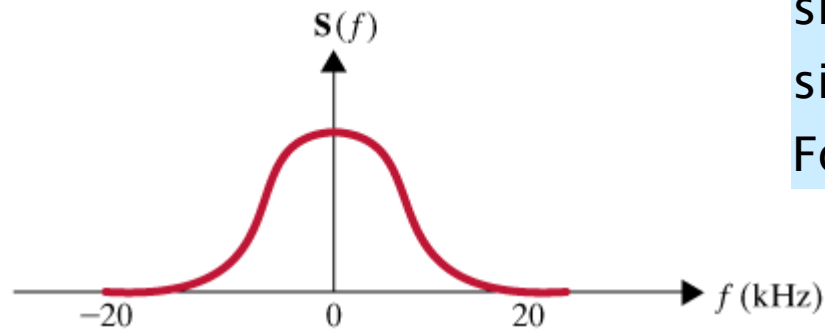
Harmonics account for 301.8W or 9.14% of the total power

$$P_{total} = 3[1000 + 0.5 + 90 + 0.1 + 10] = 3301.8 \text{ W}$$



LEARNING BY DESIGN

"Tuning-out" an AM radio station



Fourier transform of signal broadcast by two AM stations

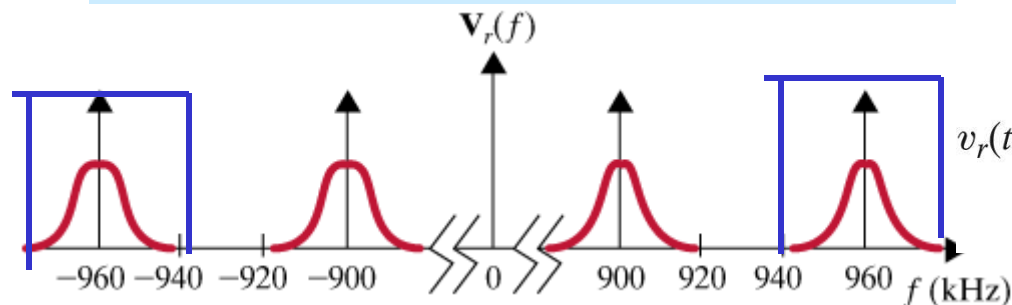
signal from station 1: $v_1(t) = [1 + s_1(t)] \cos \omega_1 t$
 signal from station 2: $v_2(t) = [1 + s_2(t)] \cos \omega_2 t$
 For simplicity assume $s_1(t) = s_2(t)$

$$f_1 = 900 \text{ kHz}, \quad f_2 = 960 \text{ kHz}$$

Assuming both stations reach the receiver with equal strength;

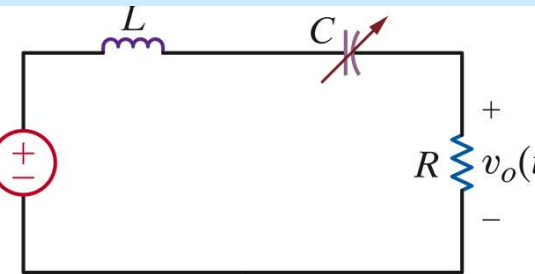
signal at antenna : $v_r(t) = K[v_1(t) + v_2(t)]$

Ideal filter to tune out one AM station



Fourier transform of received signal

Proposed tuning circuit



$$G_v(s) = \frac{\frac{sL}{R}}{s^2 + s\frac{R}{L} + \frac{1}{LC}}$$

$$G_v(s) = \frac{v_o(s)}{v_r(s)} = \frac{R}{R + Ls + \frac{1}{Cs}} \times \frac{(s/L)}{(s/L)}$$

Next we show how to design the tuning circuit by selecting suitable R, L, C

Designing the tuning circuit

$$G_v(s) = \frac{s \frac{R}{L}}{s^2 + s \frac{R}{L} + \frac{1}{LC}}$$

Design equations

center frequency: $f_o = \frac{1}{2\pi\sqrt{LC}}$

bandwidth: $BW = \frac{1}{2\pi} \frac{R}{L}$

Design specifications

$$BW \leq 60\text{kHz}$$

$$f_o = 900\text{kHz, or } 960\text{kHz}$$

More unknowns than equations. Make some choices

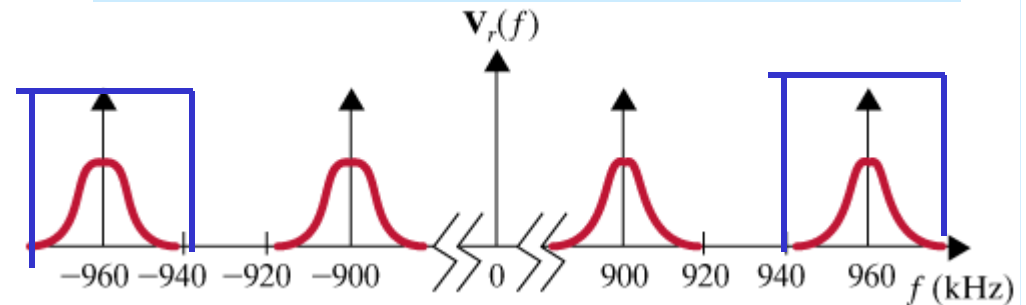
$$BW = 10\text{kHz}$$

$$R = 10\Omega$$

$$BW = \frac{1}{2\pi} \frac{R}{L} \Rightarrow L = 159.2\mu\text{H}$$

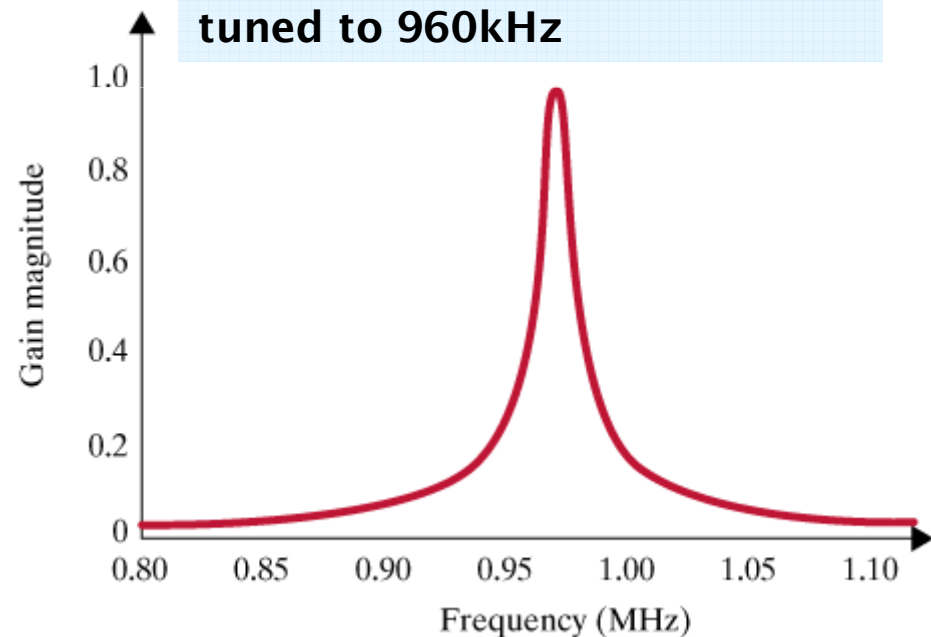
$$f_o = \frac{1}{2\pi\sqrt{LC}} \Rightarrow \begin{cases} C = 196.4\text{pF} & f_o = 900\text{kHz} \\ C = 172.6\text{pF} & f_o = 960\text{kHz} \end{cases}$$

Ideal filter to tune out one AM station



Fourier transform of received signal

Frequency response of circuit tuned to 960kHz



LEARNING EXAMPLE

An example of band-pass filter

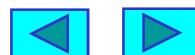
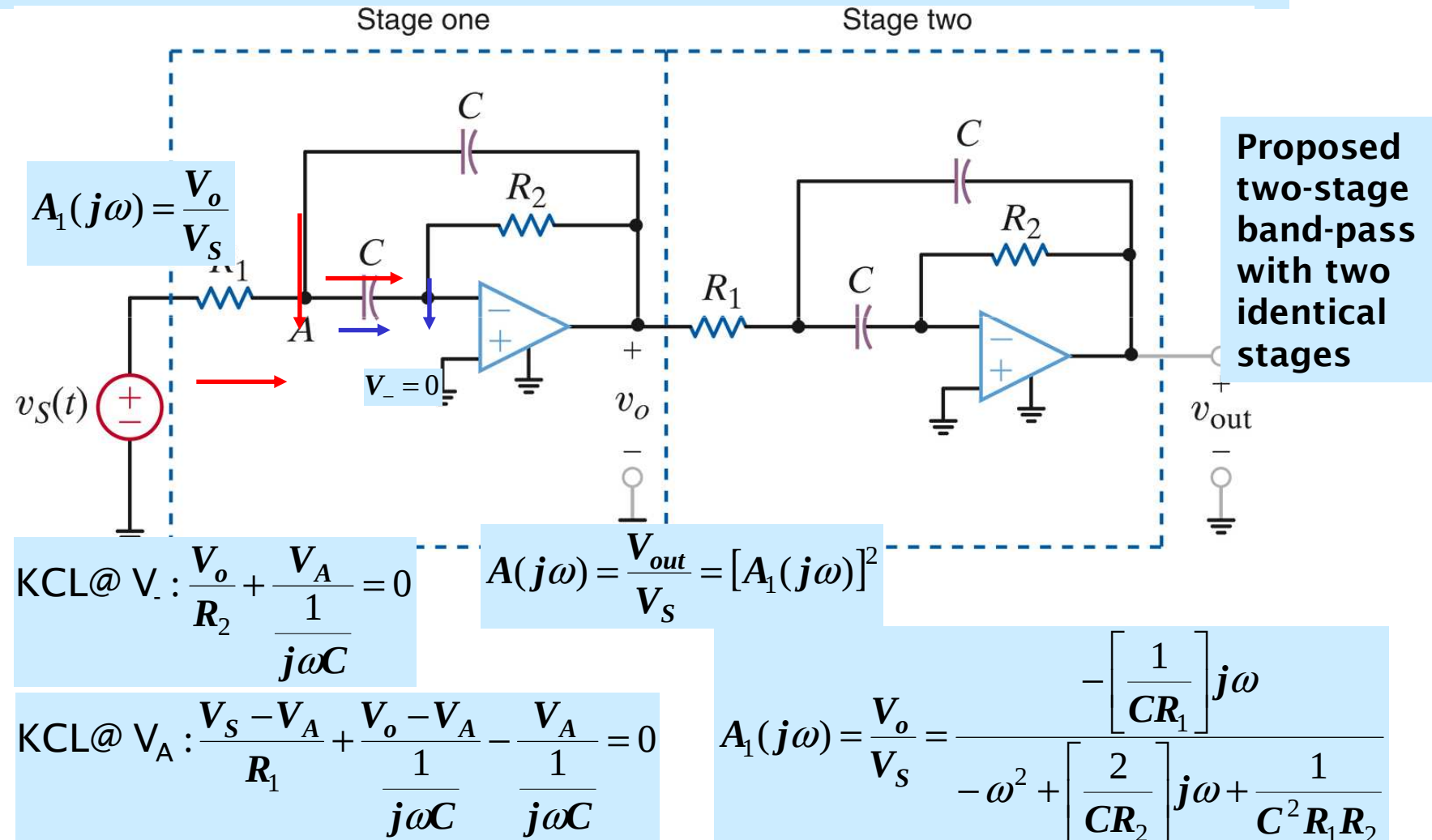
$$v_S(t) = 0.1\sin[2\pi \times 10^3 t] + 0.01\sin[2\pi \times 10^4 t] + 0.1\sin[2\pi \times 10^5 t]$$

noise

signal

noise

Design requirement: Make the signal 100 times stronger than the noise



General form of a band - pass

$$G = \frac{A_o \left[\frac{\omega_o}{Q} \right] j\omega}{-\omega^2 + \left[\frac{\omega_o}{Q} \right] j\omega + \omega_o^2}$$

ω_o = center frequency
 A_o = gain at center frequency
 Q = quality factor of the filter

In a log scale, this filter is symmetric around the center frequency. Hence, focus on 1kHz

$$A_1(j\omega) = \frac{V_o}{V_s} = \frac{-\left[\frac{1}{CR_1} \right] j\omega}{-\omega^2 + \left[\frac{2}{CR_2} \right] j\omega + \frac{1}{C^2 R_1 R_2}}$$

$$\text{Gain at } \omega = \frac{\omega_o}{10} : \frac{j \frac{A_o \omega_o^2}{10Q}}{\omega_o^2 \left(1 - \frac{1}{100}\right) + j \frac{\omega_o^2}{10Q}}$$

Equating terms one gets a set of equations that can be used for design

Design equations

$$\omega_o = \frac{1}{C \sqrt{R_1 R_2}} \quad \frac{\omega_o}{Q} = \frac{2}{CR_2}$$

$$Q = \frac{1}{2} \sqrt{\frac{R_2}{R_1}} \quad A_o = -\frac{R_2}{2R_1}$$

1kHz, 100kHz are noise frequencies
10kHz is the signal frequency

We use the requirements to constraint Q

Design constraint

$$\left| \frac{j \frac{\omega_o^2}{10Q}}{\omega_o^2 \left(1 - \frac{1}{100}\right) + j \frac{\omega_o^2}{10Q}} \right| \leq \frac{1}{\sqrt{1000}}$$

After two stages the noise gain is 1000 times smaller

Since center frequency is given this equation constrains the quality factor

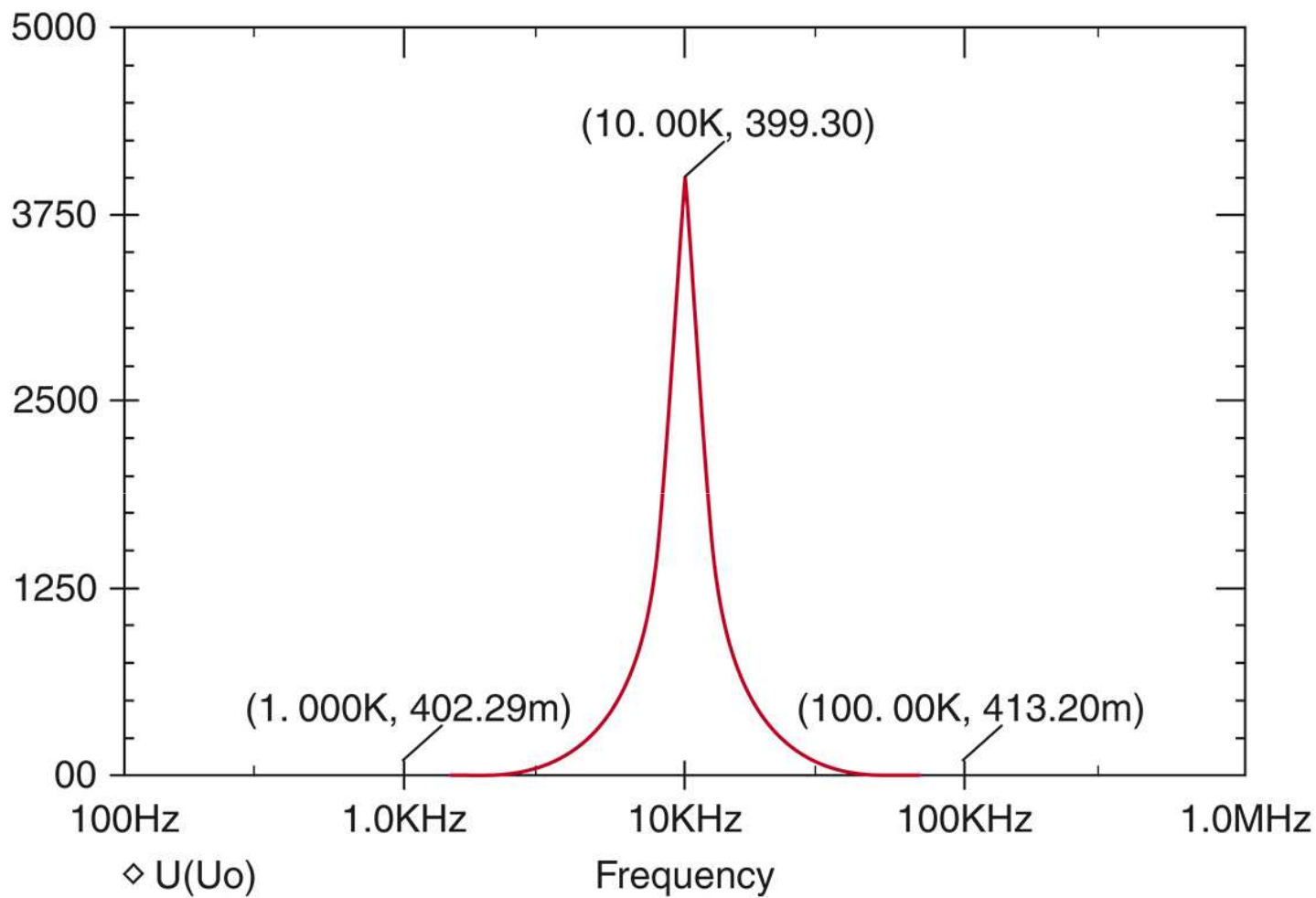
Solving for Q

$$\sqrt{1 + 100Q^2} \geq \sqrt{1000} \Rightarrow Q \approx \sqrt{10} \Rightarrow \frac{R_2}{R_1} \approx 40$$

$$\text{Picking } R_1 = 1k\Omega \Rightarrow R_2 = 40k\Omega$$

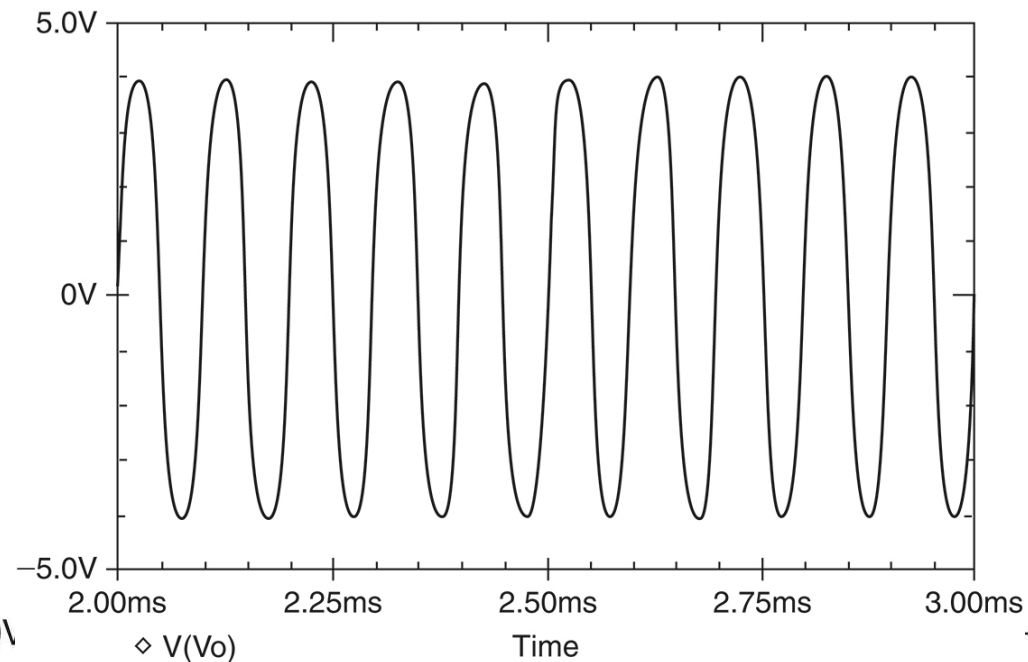
$$\omega_o = 2\pi \times 10^4 \Rightarrow C = 2.5nF$$



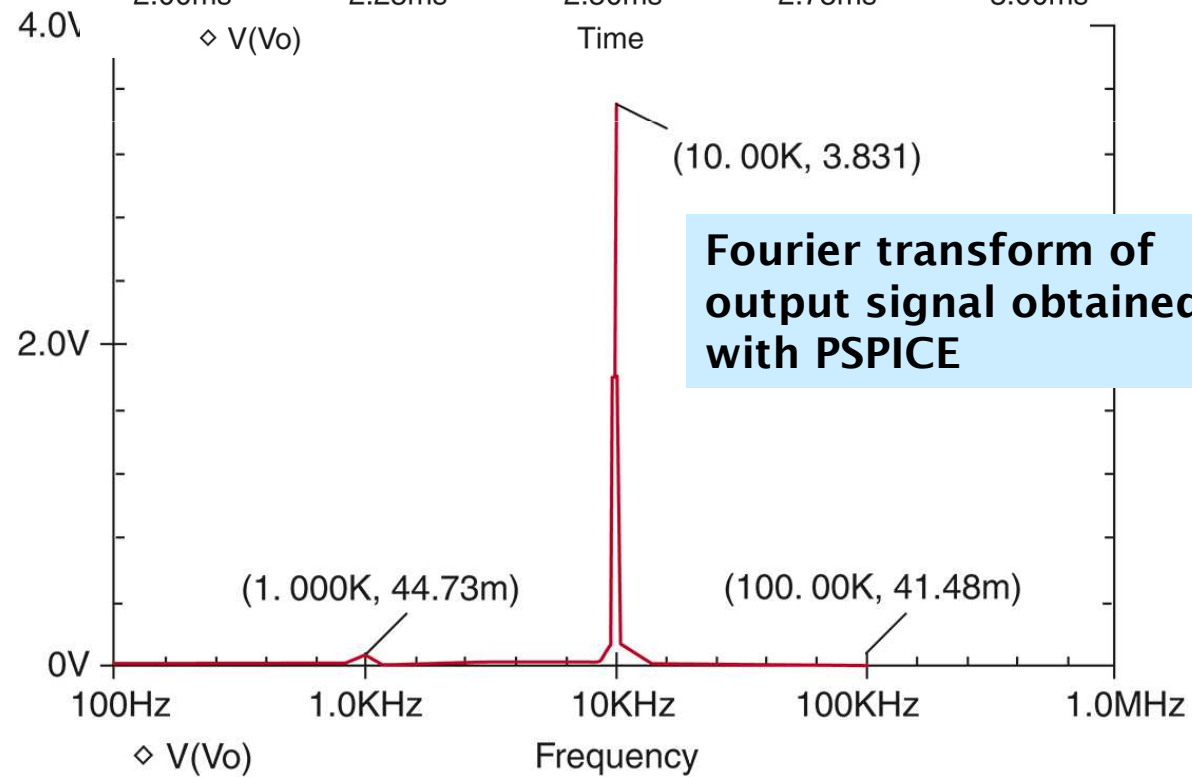


Resulting Bode plot obtained with PSPICE AC sweep

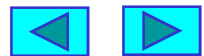




**Filter output
obtained with
PSPICE**

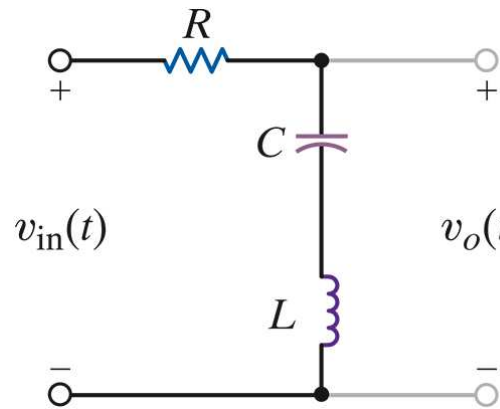


**Fourier transform of
output signal obtained
with PSPICE**



EXAMPLE

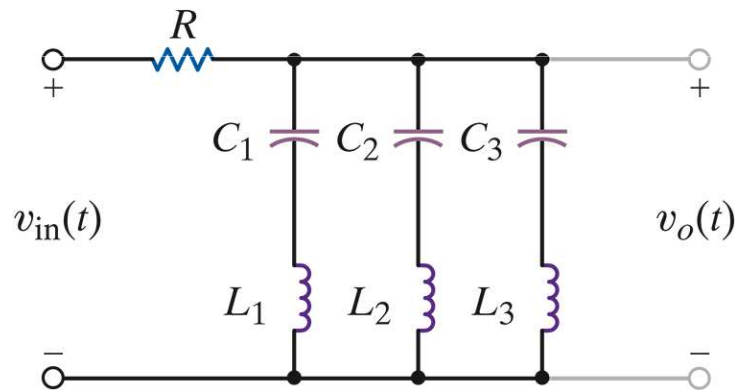
MULTIPLE FREQUENCY "TRAP" CIRCUIT



BASIC NOTCH FILTER

$$H(\omega) = \frac{V_o(\omega)}{V_{in}(\omega)} = \frac{jL\omega + \frac{1}{j\omega C}}{R + jL\omega + \frac{1}{j\omega C}} = \frac{1 - \omega^2 LC}{1 - \omega^2 LC + j\omega RC}$$

$$\omega_o = \frac{1}{\sqrt{LC}}$$



Multiple frequency trap

$$L_1 = L_2 = L_3 = 10 \mu H$$

Tuned for 10kHz

$$C_1 = \frac{1}{(2\pi)^2 f^2 L} = \frac{1}{(2\pi)^2 (10^8)(10^{-5})} = 25.3 \mu F$$

$$C_2 = \frac{1}{(2\pi)^2 (4 \times 10^8)(10^{-5})} = 6.34 \mu F$$

$$C_3 = \frac{1}{(2\pi)^2 (9 \times 10^8)(10^{-5})} = 2.81 \mu F$$

Eliminates 10kHz, 20kHz and 30kHz

Fourier

